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Discrete Mathematics
Functions

1. **Define function**.
   Solution:
   Let X and Y be any two sets. A relation f from $X\ to\ Y$ is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in f$.

2. **Check whether the following sets define a function or not?. If so, give their domain and range in each other.**
   $\{(1,(2,3)),(2,(3,4)),(3,(1,4)),(4,(1,4))\}$
   Solution:
   Let $g(1) = (2,3)$, $g(2) = (3,4), g(3) = (1,4), g(4) = (1,4)$
   Clearly each element of domain has an unique image and hence a function.
   Range is $\{(2,3),(3,4),(1,4)\}$ which is again function f is defined by
   $f(2) = 3, f(3) = 4, f(1) = 4$ with domain $\{1,2,3\}$ and range $\{3,4\}$.

3. **Determine whether $f: Z \to Z$ defined by $f(x) = x + 1$ are one to one and onto.**
   Solution:
   i)
   $$f(x) = f(y)$$
   $$x + 1 = y + 1 \Rightarrow x = y$$
   f is one to one
   ii) For every element $x \epsilon Z$ there exists an integer $x + 1$ such that $f(x + 1) = x + 1 + 1 = x + 2 \epsilon Z$
   Every element has a pre-image.
   f is onto
   from (i) and (ii) f is bijection.

4. **Define composition of function.**
   Let $X, Y\ and\ Z\ are\ sets\ and\ that\ f: X \to Y$ and $g: Y \to Z\ are$ functions. The composite relation $gof: X \to Z$ such that $(gof)(x) = g\big(f(x)\big)$ for every $x \in X$.

5. **Define identity map**
   A mapping $I_x: X \to X$ is called an identity map if $I_x = \{(x, x)/\ x\epsilon X\}$

6. **If $X = \{1, 2, 3\}$ and $f: X \to X\ and\ g: X \to X$ is given by $f = \{(1, 2), (2, 3), (3, 1)\}, g = \{(1, 2), (2, 1), (3, 3)\}$ find fog and gof.**

Solution:

$$fog = \{(1,3),(2,2),(3,1)\}$$
$$gof = \{(1,1),(2,3),(3,2)\}$$
$$\therefore\ fog \neq gof$$

7. If $f: R \to R$ and $g: R \to R$ where R is the set of real numbers. Find fog and gof if $f(x) = x^2 - 2, g(x) = x + 4$.

Solution:

$$(fog)(x) = f[g(x)] = f(x+4) = (x+4)^2 - 2 = x^2 + 16 + 8x - 2$$
$$= x^2 + 8x + 14$$
$$(gof)(x) = g[f(x)] = g(x^2 - 2) = x^2 - 2 + 4 = x^2 + 2$$

8. If $f: Z \to Z^+$ defined by $f(x) = x^2 - 2$. Find f^{-1}

Solution:

i) $f(x) = f(y) \Rightarrow x^2 - 2 = y^2 - 2 \Rightarrow x^2 = y^2$

$\Rightarrow x = y.$

$\therefore\ f$ is one to one

ii) Every element has unique pre-image.

$\therefore$ f is onto

Let $f(x) = y, y = x^2 - 2$

$x = \sqrt{y+2}$

$$\therefore f^{-1}(x) = \sqrt{x+2}$$

9. Show that $f(x) = x^3, g(x) = x^{1/3}$ for $x \in R$ are inverses of one another.

Solution:

$$(fog)(x) = f\big(g(x)\big) = f\left(x^{1/3}\right) = \left(x^{1/3}\right)^3 = x = I_x$$
$$(gof)(x) = g\big(f(x)\big) = g(x^3) = (x^3)^{1/3} = x = I_x$$
$$\therefore f = g^{-1} \ or\ g = f^{-1}$$

10. If f, g be functions from $N\ to\ N$ is the set of natural numbers so that $f(n) = n + 1, g(n) = 2n, Find\ fog\ and\ gof$.

Solution:

$$(fog)(n) = f(g(n)) = f(2n) = 2n + 1$$
$$(gof)(n) = g\big(f(n)\big) = g(n+1) = 2(n+1)$$

11. Define Commutative property

A binary operation $f: X \times X \to X$ is said to be commutative if for every $x, y \in X, f(x,y) = f(y,x)$.

12. Show that $x * y = x^y$ is a binary operation on the set of positive integers. Determine whether $*$ is commutative.

Solution:

Let $x, y \in Z^+$

$x * y = x^y \in Z^+$

$\therefore *$ is a binary operation on the set of positive integers.

$$y * x = y^x \Rightarrow x^y \neq y^x$$

$$\therefore x * y \neq y * x.$$

$*$ is not commutative.

13. Find the identity element of the group of integers with the binary operation $*$ defined by $a * b = a + b - 2$, $a, b \in Z$

Solution:

The binary operation $*$ defined by

$a * b = a + b - 2$

Let $e \in Z$ be the identity element then

$a * e = e * a = a$

$a + e - 2 = a$

$\therefore e = 2 \in Z$ *is the identity element*.

14. **What are the identity and inverse elements under $*$ defined by**

$$\boldsymbol{a * b = \frac{ab}{2}\ ,\ \ a, b \in R.}$$

Solution:

The binary operation $*$ defined as

$$a * b = \frac{ab}{2}$$

Let $e \in R$ be the identity element then

$a * e = e * a = a$

$\frac{ae}{2} = a \Rightarrow e = 2.$

$\therefore e = 2$ *is the identity element.*

Let $b \in R$ be the inverse element of $a \in R$ then

$a * b = b * a = e$

$$\frac{ab}{2} = 2 \Rightarrow b = \frac{4}{a}$$

$$\therefore b = \frac{4}{a} \text{ is the inverse of } a.$$

15. Define Characteristic function.

Let U be a universal set and A be a subset of U.

The function $\chi_A : U \to \{0,1\}$ defined by

$\chi_A(x) = \begin{cases} 1, if\ x \in A \\ 0, if\ x \notin A \end{cases}$ is called characteristic function.

16. Show that $\bar{\bar{A}} = A$ by using characteristic function.

Solution:

$\chi_{\bar{\bar{A}}}(x) = 1 - \chi_{\bar{A}}(x) = 1 - (1 - \chi_A(x)) = \chi_A(x)$

17. What is the use of Hashing function.

Hashing functions is used to generate key to a address in memory location for the files to store in random order so that files can be quickly located. The Hashing function used here is $h(k) = k\ (mod\ m)$ where m is the number of available memory locations.

18. Define Primitive Recursion function

A function is called primitive recursive iff it can be obtained from the initial functions by a finite number of operations of composition and recursion.

19. Show that the function $f(x,y) = x + y$ is a primitive recursive.

Solution:

$f(x,0) = x = U_1^1(x)$.(Using projection function)

$f(x,y+1) = x + y + 1 = S(x+y)$ (Using successer function)

$= S(f(x,y)) = S(U_3^3(x,y,f(x,y)))$ (Using projection function)

hence $f(x,y) = x + y$ is recursive and as it is obtained by initial functions by a finite number of operations of composition and recursion.

$\therefore f(x,y) = x + y$ is a primitive recursive function.

20. Define Permutation and transposition.

A bijection from a set A to itself is called a permutation of A. A cycle of length 2 is called a transposition.

21. **If $f: A \to B$ and $g: B \to C$ are mappings and $gof : A \to C$ is one-to-one, prove that f is one-to-one.**

Solution:

$$\forall x \in A, (gof)(x) = (gof)(y) \Rightarrow x = y \text{ since } gof \text{ is one to one}.$$

Consider $f(x) = f(y) \Rightarrow (gof)(x) = (gof)(y)$

$\Rightarrow x = y$

$\therefore f(x) = f(y) \Rightarrow x = y$

$Hence\ f\ is\ one\ to\ one.$

22. If A has 3 elements and B has 2 elements, how many functions are there from A to B?

Solution:

If $f: A \to B$ and A has m elements and B has n elements then there are n^m functions.

$\therefore$ There are $2^3 = 8$ functions.

23. Show that the function $f(x,y) = x\,y$ is a primitive recursive.

Solution:

$f(x,0) = 0 = Z(x) = Z(U_1^1(x))$.(Using Zero and projection function)

$f(x,y+1) = x(y+1) = xy + x = f(x,y) + x$

$= U_3^3(x, y, f(x, y)) + U_3^1(x, y, f(x, y))$ (Using projection function)
$= f_1(U_3^3(x, y, f(x, y)), U_3^1(x, y, f(x, y)))$ (Using $f_1(x, y) = x + y$)
$f(x, y) = xy$ is recursive since $f_1(x, y) = x + y$ is primitive recursive and as it is obtained by initial functions by a finite number of operations of composition and recursion.
$\therefore f(x, y) = xy$ is a primitive recursive function.

24. Show that the function $f(x, y) = x - y$ is a partial recursive.
Solution:
Clearly , $y \in N$, the function is well defined only for $x > y$. Therefore, $f(x, y) = x - y$ for only $x > y$ is partial recursive.

25. Let $h(x, y) = g(f_1(x, y)), f_2(x, y))$ for all positive integers x and y where $f_1(x, y) = x^2 + y^2, f_2(x, y) = x$ and $(x, y) = xy^2$. Find $h(x, y)$ interms of x any y.
Solution: $h(x, y) = g(f_1(x, y)), f_2(x, y))$
$h(x, y) = g(x^2 + y^2, x) = (x^2 + y^2)x^2$

26. Prove by an example, composition of function is not commutative.
Solution:
Let $f: R \to R$ and $g: R \to R$ and let $f(x) = x^2$ and $g(x) = 2x$
Then $(fog)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$
$(gof)(x) = g(f(x)) = g(x^2) = 2x^2$
Hence $fog \neq gof$
$\therefore$ Composition of function is not commutative.

27. Define Cyclic permutation
Let $b_1, b_2, \dots, b_r$ be r distinct elements of the set A . The permutation $P: A \to A$ defined by $P(b_1) = b_2, P(b_2) = b_3, \dots, P(b_{r-1}) = b_r, P(b_r) = b_1$ is called a cyclic permutation of length r, or simply cycle of length r and it will denoted by $(b_1, b_2, \dots, b_r)$.

28. Show that the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 1 & 3 \end{pmatrix}$ is odd.
Solution:
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 2 & 4 & 1 & 3 \end{pmatrix} = (1\ 5)\ (2\ 6\ 3) = (1\ 5)\ (2\ 6)\ (2\ 3)$
The given permutation can be expressed as the product of an odd number of transpositions and hence the permutation is odd.

29. If $A = (1\ 2\ 3\ 4\ 5), B = (2\ 3)\ (4\ 5)$. Find AoB
Solution:

$$AoB = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} o \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} = (1\ 2\ 4)$$

30. Show that the permutation $\begin{pmatrix} 1&2&3&4&5&6 \\ 3&1&2&5&6&4 \end{pmatrix}$ is even.

Solution:

$\begin{pmatrix} 1&2&3&4&5&6 \\ 3&1&2&5&6&4 \end{pmatrix} = (1\ \ 3\ \ 2)\,(4\ \ 5\ \ 6) = (1\ \ 3)\,(1\ \ 2)\,(4\ \ 5)(4\ \ 6)$

The given permutation can be expressed as the product of even number of transpositions and hence the permutation is even.

PART B

31. i) Find all mappings from $A = \{1, 2, 3\}$ *to* $B = \{4, 5\}$. Find which of them are one-to one and which are onto.

Solution:

All possible mappings from A *to* B are given below

a) $\{(1,4), (2,4), (3,4)\}$
b) $\{(1,5), (2,5), (3,5)\}$
c) $\{(1,4), (2,4), (3,5)\}$
d) $\{(1,4), (2,5), (3,4)\}$
e) $\{(1,5), (2,4), (3,4)\}$
f) $\{(1,4), (2,5), (3,5)\}$
g) $\{(1,5), (2,4), (3,5)\}$
i) $\{(1,5), (2,5), (3,4)\}$

Here all the mappings are not one to one functions because at least one element of A is mapped to more than one element of B.

Here all the mappings are not onto functions because every element of B has pre image in A but it is not unique.

ii) If $f = \begin{pmatrix} 1&2&3&4 \\ 3&2&1&4 \end{pmatrix}$ and $g = \begin{pmatrix} 1&2&3&4 \\ 2&3&4&1 \end{pmatrix}$ are permutations, prove that $(g\ o\ f)^{-1} = f^{-1} o\ g^{-1}$

Solution:

$f^{-1}(1) = 3, f^{-1}(2) = 2, f^{-1}(3) = 1, f^{-1}(4) = 4, g^{-1}(1) = 4, g^{-1}(2) = 1,$
$g^{-1}(3) = 2, g^{-1}(4) = 3$

$f^{-1} = \begin{pmatrix} 1&2&3&4 \\ 3&2&1&4 \end{pmatrix}, g^{-1} = \begin{pmatrix} 1&2&3&4 \\ 4&1&2&3 \end{pmatrix}$

$(g\ o\ f)(1) = g(f(1)) = g(3) = 4, (g\ o\ f)(2) = g(f(2)) = g(2) = 3,$
$(g\ o\ f)(3) = g(f(3)) = g(1) = 2, (g\ o\ f)(4) = g(f(4)) = g(4) = 1$

$g\ o\ f = \begin{pmatrix} 1&2&3&4 \\ 4&3&2&1 \end{pmatrix}$

$(g\ o\ f)^{-1}(1) = 4, (g\ o\ f)^{-1}(2) = 3, (g\ o\ f)^{-1}(3) = 2, (g\ o\ f)^{-1}(4) = 1$

$(g\ o\ f)^{-1} = \begin{pmatrix} 1&2&3&4 \\ 4&3&2&1 \end{pmatrix} \dots (1)$

$(f^{-1} o\ g^{-1})(1) = f^{-1}(\ g^{-1}(1)) = f^{-1}(4) = 4,$
$(f^{-1} o\ g^{-1})(2) = f^{-1}(\ g^{-1}(2)) = f^{-1}(1) = 3,$
$(f^{-1} o\ g^{-1})(3) = f^{-1}(\ g^{-1}(3)) = f^{-1}(2) = 2,$
$(f^{-1} o\ g^{-1})(4) = f^{-1}(\ g^{-1}(4)) = f^{-1}(3) = 1$

$f^{-1} o\ g^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \ldots (2)$

From (1) and (2), we get

$(g\ o\ f)^{-1} = f^{-1} o\ g^{-1}$

iii) If R denotes the set of real numbers and $f: R \to R$ is given by $f(x) = x^3 - 2$, find f^{-1}.

Solution:

To prove f is one to one:

$\forall x, y \in R\ let\ f(x) = f(y) \Rightarrow x^3 - 2 = y^3 - 2 \Rightarrow x^3 = y^3 \Rightarrow x = y$

$\therefore$ f is one to one

To prove f is onto:

$$y = x^3 - 2 \Rightarrow x^3 = y + 2 \Rightarrow x = (y+2)^{1/3} \in R$$

$$\forall x \in R, x = f((x+2)^{1/3})$$

$$\therefore\ \forall x \in R, there\ is\ a\ pre\ image\ (x+2)^{1/3} \in R$$

Every element has unique pre-image

$\therefore$ f is onto

$\therefore$ f is bijection $\Rightarrow$ f^{-1} exists.

Let $y = f(x) \Rightarrow x = f^{-1}(y)$

$$y = x^3 - 2 \Rightarrow x^3 = y + 2 \Rightarrow x = f^{-1}(y) = (y+2)^{1/3}$$

$$f^{-1}(x) = (x+2)^{1/3}$$

32. i)If Z^+ denote the set of positive integers and Z denote the set of integers. Let $f: Z^+ \to Z$ be defined by

$f(n) = \begin{cases} \frac{n}{2}, if\ n\ is\ even \\ \frac{1-n}{2}, if\ n\ is\ odd \end{cases}$. Prove that f is a bijection and find f^{-1}.

Solution:

To prove f is one to one:

$$\forall x, y \in Z^+$$

Case: 1 when $x\ and\ y\ are\ even$

$f(x) = f(y) \Rightarrow \frac{x}{2} = \frac{y}{2} \Rightarrow x = y \ldots (1)$

Case: 2 when $x\ and\ y\ are\ odd$

$f(x) = f(y) \Rightarrow \frac{1-x}{2} = \frac{1-y}{2} \Rightarrow 1 - x = 1 - y \Rightarrow x = y \ldots (2)$

From (1) and (2), we get

$\therefore$ f is one to one

To prove f is onto:

$When\ x\ is\ even$

Let $y = \frac{x}{2} \Rightarrow x = 2y$

$$\forall x \in Z, x = f(2x)$$

$$\therefore\ \forall x \in Z, there\ is\ a\ pre\ image\ 2x \in\ Z^+$$

$When\ x\ is\ odd$
Let $y = \frac{1-x}{2} \Rightarrow 1 - x = 2y \Rightarrow x = 1 - 2y$

$$\forall x \in Z, x = f(1-2x)$$

$$\therefore\ \forall x \in Z, there\ is\ a\ pre\ image\ 1-2x \in\ Z^+$$

Every element has unique pre-image
$\therefore\ f$ is onto
$\therefore\ f$ is bijection $\Rightarrow\ f^{-1}$ exists.

$When\ x\ is\ even$
Let $y = \frac{x}{2} \Rightarrow x = f^{-1}(y) = 2y$
$When\ x\ is\ odd$
Let $y = \frac{1-x}{2} \Rightarrow 1 - x = 2y \Rightarrow x = f^{-1}(y) = 1 - 2y$

$$f^{-1}(n) = \begin{cases} 2n, if\ n\ is\ even \\ 1-2n, if\ n\ is\ odd \end{cases}$$

ii)If A, B and C be any three nonempty sets. Let $f : A \to B$ and $g : B \to C$ be mappings. If f and g are onto, prove that $gof : A \to C$ is onto. Also give an example to show that gof may be onto but both f and g need not be onto.
Solution:
Since $f : A \to B$ is onto
$f(x) = y, \forall x \in A\ and\ y \in B$ … (1)
Since $g : B \to C$ is onto
$g(y) = z, \forall z \in C\ and\ y \in B$ … (2)
$\forall x \in A, gof(x) = g(f(x)) = g(y) = z\ [from\ (1)\ and\ (2)]$
$\therefore \forall z \in C\ there\ exists\ a\ preimage\ x \in A\ such\ that\ gof(x) = z$
$\therefore gof : A \to C$ is onto
For example
Let $A = \{1, 2\}, B = \{a, b, c\} and\ C = \{d, e\}$

$$f = \{(1, a), (2, b)\}, g = \{(a, d), (b, e), (c, e)\}$$

$$gof(1) = g(f(1)) = g(a) = d$$

$$gof(2) = g(f(2)) = g(b) = e$$

$$gof = \{(1, d), (2, e)\}$$

The function f is not onto because $c \in B$ does not have pre image.
The function g is not onto because every element of C have pre image but it is not unique. $e \in C\ have\ two\ pre\ images\ b, c \in B$
The function gof is onto because every element of C have pre image and it is unique.

33. i)If the function f and g be defined $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Determine the composition function fog and gof.

Solution:

$fog(x) = f(g(x)) = f(x^2 - 2) = 2(x^2 - 2) + 1 = 2x^2 - 3$

$gof(x) = g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2 = 4x^2 + 4x + 1 - 2$

$gof(x) = 4x^2 + 4x - 1$

ii) Let a and b be any positive integers and suppose Q is defined recursively as follows:

$\boldsymbol{Q(a,b)} = \begin{cases} \mathbf{0}, & \boldsymbol{if\ a} < b \\ \boldsymbol{Q(a-b,b)+1}, & \boldsymbol{if\ b \le a} \end{cases}$ **. Find $\boldsymbol{Q(2,5)}$, $\boldsymbol{Q(12,5)}$, $\boldsymbol{Q(5861,7)}$.**

Solution:

$Q(2,5) = 0 \ since\ 2 < 5$

$Q(12,5) = Q(12 - 5,5) + 1 = Q(7,5) + 1$

$\quad = (Q(7 - 5,5) + 1) + 1 = Q(2,5) + 2 = 0 + 2 = 2$

$Q(5861,7) = Q(5861 - 7,7) + 1 = Q(5854,7) + 1$

$= Q(5847,7) + 2 = Q(5840,7) + 3 = Q(5833,7) + 4 = Q(5826,7) + 5$

$= Q(5819,7) + 6 = Q(5812,7) + 7 = Q(5805,7) + 8$ ...

$= Q(2,7) + 837 = 837$

$This\ function\ is\ the\ quotient\ of\ a\ divided\ by\ b$

$\therefore \frac{5861}{7} = 837.28$

$\therefore Q(5861,7) = 837$

34. i) If $\boldsymbol{f: R \to R}$ be defined by $\boldsymbol{f(x) = 2x - 3}$. Find a formula for $\boldsymbol{f^{-1}}$.

Solution:

To prove f is one to one:

$\forall x, y \in R\ let\ f(x) = f(y) \Rightarrow 2x - 3 = 2y - 3 \Rightarrow 2x = 2y \Rightarrow x = y$

$\therefore$ f is one to one

To prove f is onto:

$$y = 2x - 3 \Rightarrow 2x = y + 3 \Rightarrow x = \frac{y+3}{2} \in R$$

$$\forall x \in R, x = f\left(\frac{x+3}{2}\right)$$

$$\therefore \ \forall x \in R, there\ is\ a\ pre\ image\ \frac{x+3}{2} \in R$$

Every element has unique pre-image

$\therefore$ f is onto

$\therefore$ f is bijection $\Rightarrow$ f^{-1} exists.

Let $y = f(x) = 2x - 3 \Rightarrow 2x = y + 3 \Rightarrow x = f^{-1}(y) = \frac{y+3}{2}$

$$f^{-1}(x) = \frac{x+3}{2}$$

ii) Show that $\boldsymbol{A \cap (BUC) = (A \cap B)\ U\ (A \cap C)}$ by using Characteristic functions.

Solution:

$\chi_{A\cap(BUC)}(x) = 1 \Leftrightarrow x \in A \cap (BUC)$
$\Leftrightarrow x \in A \text{ and } x \in (BUC)$
$\Leftrightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$
$\Leftrightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$
$\Leftrightarrow x \in A \cap B \text{ or } x \in A \cap C$
$\Leftrightarrow x \in (A \cap B)\, U\, (A \cap C)$
$\Leftrightarrow \chi_{(A\cap B)\, U\, (A\cap C)}(x) = 1 \dots (1)$

$\chi_{A\cap(BUC)}(x) = 0 \Leftrightarrow x \notin A \cap (BUC)$
$\Leftrightarrow x \notin A \text{ and } x \notin (BUC)$
$\Leftrightarrow x \notin A \text{ or } (x \notin B \text{ and } x \notin C)$
$\Leftrightarrow (x \notin A \text{ or } x \notin B) \text{ and } (x \notin A \text{ or } x \notin C)$
$\Leftrightarrow x \notin A \cap B \text{ and } x \notin A \cap C$
$\Leftrightarrow x \notin (A \cap B)\, U\, (A \cap C)$
$\Leftrightarrow \chi_{(A\cap B)\, U\, (A\cap C)}(x) = 0 \dots (2)$

From (1) *and* (2) *we get*
$\chi_{A\cap(BUC)}(x) = \chi_{(A\cap B)\, U\, (A\cap C)}(x)$
$\therefore A \cap (BUC) = (A \cap B)\, U\, (A \cap C)$.

Aliter:

$\chi_{A\cap(BUC)}(x) = \chi_A(x)\chi_{BUC}(x)$
$= \chi_A(x)\big(\chi_B(x) + \chi_C(x) - \chi_{B\cap C}(x)\big)$
$= \chi_A(x)\big(\chi_B(x) + \chi_C(x) - \chi_B(x)\chi_C(x)\big)$
$= \chi_A(x)\chi_B(x) + \chi_A(x)\chi_C(x) - \chi_A(x)\chi_B(x)\chi_C(x)$
$= \chi_{A\cap B}(x) + \chi_{A\cap C}(x) - \chi_{A\cap B\cap C}(x)$
$= \chi_{A\cap B}(x) + \chi_{A\cap C}(x) - \chi_{(A\cap B)\cap(A\cap C)}(x)$
$= \chi_{(A\cap B)\, U\, (A\cap C)}(x)$

35. i)Show that $f: R - \{3\} \to R - \{1\}$ given by $f(x) = \frac{x-2}{x-3}$ is a bijection.

Solution:

To prove f is one to one:

$\forall x, y \in R - \{3\}, let\ f(x) = f(y) \Rightarrow \dfrac{x-2}{x-3} = \dfrac{y-2}{y-3}$

$\Rightarrow (x-2)(y-3) = (y-2)(x-3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6 \Rightarrow 3y - 2y = 3x - 2x \Rightarrow x = y$

$\therefore$ f is one to one

To prove f is onto:

$y = \dfrac{x-2}{x-3} \Rightarrow y(x-3) = x - 2 \Rightarrow yx - 3y = x - 2 \Rightarrow yx - x = 3y - 2$

$\Rightarrow x(y-1) = 3y - 2 \Rightarrow x = \dfrac{3y-2}{y-1} \in R - \{1\}$

$$\forall x \in R - \{1\}, x = f\left(\frac{3x-2}{x-1}\right)$$

$$\therefore \ \forall x \in R - \{1\}, there\ is\ a\ pre\ image\ \frac{3x-2}{x-1} \in R - \{3\}$$

Every element has unique pre-image

$\therefore$ f is onto

$\therefore$ f is bijection $\Rightarrow$ f^{-1} exists.

Let $y = f(x) = \frac{x-2}{x-3} \Rightarrow y(x-3) = x - 2 \Rightarrow yx - 3y = x - 2$

$$\Rightarrow yx - x = 3y - 2$$

$$\Rightarrow x(y-1) = 3y - 2 \Rightarrow x = f^{-1}(y) = \frac{3y-2}{y-1} \in R - \{1\}$$

$$f^{-1}(x) = \frac{3x-2}{x-1}$$

ii) Let $f(x) = x + 2$, $g(x) = x - 2$ and $h(x) = 3x$ for $x \in R$. Find gof and $fo(goh)$.

Solution:

$(gof)(x) = g\big(f(x)\big) = g(x+2) = x + 2 - 2 = x$

$\big(fo(goh)\big)(x) = f\big((goh)(x)\big) = f\Big(g\big(h(x)\big)\Big) = f\big(g(3x)\big) = f(3x-2)$

$\big(fo(goh)\big)(x) = 3x - 2 + 2 = 3x$

iii)Let $D(x)$ denote the number of divisors of x. Show that $D(x)$ is a primitive function.

Proof:

Let $r(m, n)$ denote the remainder got when n is divided by m.

If m is a divisor of n, then

$$r(m, n) = 0$$

$$\overline{sg}\{r(m, n)\} = 1$$

Hence, the number of divisors of n, say $f(n)$ is given by

$$f(n) = \sum_{m=1}^{n} \overline{sg}\{r(m, n)\} \ldots (1)$$

$\overline{sg}\{r(m, n)\}$ is the composition of two primitive recursive functions and hence, primitive recursive.

Since $f(n)$ is the sum of finite number of primitive recursive functions, it is also a primitive recursive.

[For example, the divisors of 6 are 1, 2, 3 and 6

The number of divisors of 6 are 4

Also $\overline{sg}\{r(1,6)\} + \overline{sg}\{r(2,6)\} + \overline{sg}\{r(3,6)\} + \overline{sg}\{r(4,6)\}$

$+\overline{sg}\{r(5,6)\} + \overline{sg}\{r(6,6)\} = 1 + 1 + 1 + 0 + 0 + 1 = 4$]

36. i) Prove that $Pr(x)$, the odd and even parity function is primitive recursive.

Solution:

By definition $Pr(x) = \begin{cases} 0, & if\ x = 0\ or\ even \\ 1, & if\ x\ is\ odd \end{cases}$

$$Pr(0) = 0 = Z(x) \dots (1)$$

$$Pr(x+1) = \begin{cases} 1, & if\ x = 0\ or\ even \\ 0, & if\ x\ is\ odd \end{cases}$$

$$= \overline{sg}\{Pr(x)\} = \overline{sg}\{U_2^2\{x, Pr(x)\}\}$$

Thus, $Pr(x)$ is defined recursively from the initial functions $Z(x), \overline{sg}(x)$, and U_2^2 using composition.

$\therefore$ $Pr(x)$ is primitive recursive.

ii)Define even and odd permutations. Show that the permutations $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 8 & 6 & 1 & 4 & 3 \end{pmatrix}$ *and* $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$ are respectively even and odd.

Solution:

A Permutation function is called even if the number of its transposition is even.

A Permutation function is called odd if the number of its transposition is odd.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 7 & 8 & 6 & 1 & 4 & 3 \end{pmatrix} = (1\ 2\ 5\ 6)(3\ 7\ 4\ 8)$$

$$= (1\ 2)(1\ 5)(1\ 6)(3\ 7)(3\ 4)(3\ 8)$$

There are 6 transposition in f.

$\therefore$ f is even permutation

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix} = (1\ 4\ 2\ 3) = (1\ 4)(1\ 2)(1\ 3)$$

There are 3 transposition in f.

$\therefore$ f is odd permutation

37. i) If f and g are bijection on a set A, prove that fog is also bijection.

Solution:

Let A, B and C be any three nonempty sets. Let $f : A \to B$ and $g : B \to C$ be then $gof : A \to C$.

To prove $gof : A \to C$ is one to one:

$\forall x, y \in A, gof(x) = gof(y)$

$\Rightarrow g(f(x)) = g(f(y))$

$\Rightarrow f(x) = f(y)$ $[Since\ g\ is\ one\ to\ one]$

$\Rightarrow x = y$ $[Since\ f\ is\ one\ to\ one]$

$\forall x, y \in A, gof(x) = gof(y) \Rightarrow x = y$

$\therefore$ $gof : A \to C$ is one to one

To prove $gof : A \to C$ is onto:

Since $f : A \to B$ is onto

$f(x) = y, \forall x \in A\ and\ y \in B \dots (1)$

Since $g : B \to C$ is onto

$g(y) = z, \forall z \in C\ and\ y \in B \dots (2)$

$\forall x \in A, gof(x) = g\big(f(x)\big) = g(y) = z\ [from\ (1)\ and\ (2)]$
$\therefore \forall z \in C\ there\ exists\ a\ preimage\ x \in A\ such\ that\ gof(x) = z$
$\therefore gof : A \to C$ is onto
$\therefore gof : A \to C$ is bijection.

ii) If $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 3 & 1 \end{pmatrix}$ permutations on the set $A = \{1, 2, 3, 4, 5\}$. Find the permutation g on A such that $fog = hof$.

Solution:
Given that $fog = hof \Rightarrow g = f^{-1}\, o(hof)$
$f^{-1}(1) = 4, f^{-1}(2) = 1, f^{-1}(3) = 5, f^{-1}(4) = 2, f^{-1}(5) = 3$
$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 2 & 3 \end{pmatrix}$
$(h\ o\ f)(1) = h\big(f(1)\big) = h(2) = 2, (h\ o\ f)(2) = h\big(f(2)\big) = h(4) = 3,$
$(h\ o\ f)(3) = h\big(f(3)\big) = h(5) = 1, (h\ o\ f)(4) = h\big(f(4)\big) = h(1) = 5,$
$(h\ o\ f)(5) = h\big(f(5)\big) = h(3) = 4$
$h\ o\ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}$
$\big(f^{-1}o(h\ o\ f)\big)(1) = f^{-1}\big((h\ o\ f)(1)\big) = f^{-1}(2) = 1$
$\big(f^{-1}o(h\ o\ f)\big)(2) = f^{-1}\big((h\ o\ f)(2)\big) = f^{-1}(3) = 5$
$\big(f^{-1}o(h\ o\ f)\big)(3) = f^{-1}\big((h\ o\ f)(3)\big) = f^{-1}(1) = 4$
$\big(f^{-1}o(h\ o\ f)\big)(4) = f^{-1}\big((h\ o\ f)(4)\big) = f^{-1}(5) = 3$
$\big(f^{-1}o(h\ o\ f)\big)(5) = f^{-1}\big((h\ o\ f)(5)\big) = f^{-1}(4) = 2$
$\therefore g = f^{-1}\, o(hof) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix}$

iii)The Ackerman function $A(x, y)$ is defined by $A(0, y) = y + 1$; $A(x + 1, 0) = A(x, 1)$; $A(x + 1, y + 1) = A(x, A(x + 1, y))$. Find $A(2, 1)$.

Solution:
$A(0, y) = y + 1$ … (1)
$A(x + 1,0) = A(x, 1)$ … (2)
$A(x + 1, y + 1) = A\big(x, A(x + 1, y)\big)$ … (3)
$A(2,1) = A\big(1, A(2,0)\big)$ … (4) $[from\ (3)]$
$A(2,0) = A(1,1)$ … (5)$[from\ (2)]$
$A(1,1) = A\big(0, A(1,0)\big)$ … (6)$[from\ (3)]$
$A(1,0) = A(0,1) = 1 + 1 = 2$ … (7)$[from\ (2) \&\ (1)]$
$A(1,1) = A\big(0, A(1,0)\big) = A(0,2) = 2 + 1 = 3$ … (8)$[from\ (3), (6) \&\ (7)]$
$A(2,0) = A(1,1) = 3$ … (9)$[from\ (5) \&\ (8)]$
$A(2,1) = A\big(1, A(2,0)\big) = A(1,3)$ … (10)$[from\ (4) \&\ (9)]$
$A(1,3) = A\big(0, A(1,2)\big)$ … (11) $[from\ (3)]$
$A(1,2) = A\big(0, A(1,1)\big) = A(0,3) = 3 + 1 = 4$ … (12) $[from\ (3)\ \&\ (1)]$
$A(1,3) = A\big(0, A(1,2)\big) = A(0,4) = 4 + 1 = 5$ … (13)$[from\ (11)\ \&\ (12)]$
$A(2,1) = 5\ [from\ (10)\ \&\ (13)]$

38. i) Let $a < b$. If $f: [a, b] \to [0, 1]$ is defined by $f(x) = \frac{x-a}{b-a}$, Prove that f is a bijection and find its inverse.

Solution:

To prove f is one to one:

$$\forall x, y \in [a, b], let\ f(x) = f(y) \Rightarrow \frac{x-a}{b-a} = \frac{y-a}{b-a}$$
$$\Rightarrow x - a = y - a \Rightarrow x = y$$

$\therefore$ f is one to one

To prove f is onto:

$$y = \frac{x-a}{b-a} \Rightarrow y(b-a) = x - a \Rightarrow x = y(b-a) + a \in [\boldsymbol{a}, \boldsymbol{b}]$$
$$\forall x \in [\mathbf{0}, \mathbf{1}], x = f(x(b-a) + a)$$
$$\therefore \ \forall x \in [\mathbf{0}, \mathbf{1}], there\ is\ a\ pre\ image\ x(b-a) + a \in [\boldsymbol{a}, \boldsymbol{b}]$$

Every element has unique pre-image

$\therefore$ f is onto

$\therefore$ f is bijection $\Rightarrow$ f^{-1} exists.

Let $y = f(x) == \frac{x-a}{b-a} \Rightarrow y(b-a) = x - a$

$$\Rightarrow x = f^{-1}(y) = y(b-a) + a \in [\boldsymbol{a}, \boldsymbol{b}]$$
$$f^{-1}(x) = x(b-a) + a$$

ii) If $f: A \to B$ and $g: B \to C$ are mappings such that $gof : A \to C$ is bijection prove that g is onto and f is one to one.

Solution:

To prove f is one to one:

$$\forall x, y \in A, Let\ us\ assume\ that\ f(x) = f(y)$$
$$\Rightarrow g(f(x)) = g(f(y))$$
$$\Rightarrow gof(x) = gof(y) \Rightarrow x = y\ [Since\ gof\ is\ one\ to\ one]$$
$$\therefore f(x) = f(y) \Rightarrow x = y$$

$\therefore$ f is one to one

To prove g is onto:

$\forall z \in C\ , gof(x) = z\ Since\ gof\ is\ onto\ \forall x \in A$

$\Rightarrow g\big(f(x)\big) = z\ \Rightarrow g(y) = z, where\ y = f(x) \in B$

$\therefore \forall z \in C\ \ there\ exists\ y \in B\ such\ that\ g(y) = z$

$\therefore g\ is\ onto$

39. i)Prove that composition of functions is associative.

Proof: Let $f: A \to B, g: B \to C\ and\ h: C \to D$ are the functions then

$gof: A \to C, ho(gof): A \to D$ ... (1)

$hog: B \to D, (hog)of: A \to D$ ... (2)

$\forall x \in A,$

$ho(gof)(x) = h\big((gof)(x)\big) = h\Big(g\big(f(x)\big)\Big) = hog\big(f(x)\big)$

$$= \big((hog)of\big)(x) \ldots (3)$$

$From\ (1), (2)\ and\ (3), we\ get$

Composition of functions is associative

ii)Prove that $(g\ o\ f)^{-1} = f^{-1} o\ g^{-1}$ where $f: X \to Y$ and $g: Y \to Z$ be two invertible functions.

Proof:

If $f: X \to Y\ and\ g: Y \to Z$ are invertible then $f\ and\ g\ are\ bijection\ then$

$$g\ o\ f: X \to Z\ is\ also\ bijection. \therefore\ gof\ is\ invertible.$$

$\therefore (g\ o\ f)^{-1}: Z \to X \ldots (1)$

$f^{-1}: Y \to X, g^{-1}: Z \to Y \Rightarrow f^{-1} o\ g^{-1}: Z \to X \ldots (2)$

$$\big((g\ o\ f) o (f^{-1} o\ g^{-1})\big)(x) = (g\ o\ (f o f^{-1}) o\ g^{-1})(x) = \big(g\ o\ I_y o\ g^{-1}\big)(x)$$
$$= (g\ o\ g^{-1})(x) = I_z(x) \ldots (3)$$
$$\big((f^{-1} o\ g^{-1}) o (g\ o\ f)\big)(x) = (f^{-1}\ o\ (g^{-1} o g) o\ f)(x) = \big(f^{-1}\ o\ I_y o\ f\big)(x)$$
$$= (f^{-1}\ o\ f)(x) = I_x(x) \ldots (4)$$

$From\ (1), (2), (3)\ and\ (4), we\ get$

$(g\ o\ f)^{-1} = f^{-1} o\ g^{-1}$

Aliter:

If $f: X \to Y\ and\ g: Y \to Z$ are invertible then $f\ and\ g\ are\ bijection\ then$

$$g\ o\ f: X \to Z\ is\ also\ bijection. \therefore\ gof\ is\ invertible.$$

$\therefore (g\ o\ f)^{-1}: Z \to X \ldots (1)$

$f^{-1}: Y \to X, g^{-1}: Z \to Y \Rightarrow f^{-1} o\ g^{-1}: Z \to X \ldots (2)$

Now for any $x \in X, let\ f(x) = y\ and\ g(y) = z$

$$gof(x) = g\big(f(x)\big) = g(y) = z$$
$$(g\ o\ f)^{-1}(z) = x \ldots (3)$$
$$f(x) = y \Rightarrow f^{-1}(y) = x$$
$$g(y) = z \Rightarrow g^{-1}(z) = y$$
$$f^{-1} o\ g^{-1}(z) = f^{-1}\big(g^{-1}(z)\big) = f^{-1}(y) = x \ldots (4)$$

$From\ (1), (2), (3)\ and\ (4), we\ get$

$(g\ o\ f)^{-1} = f^{-1} o\ g^{-1}$

40. **i)Using Characteristic function, Prove that $|A \cup B| = |A| + |B| - |A \cap B|$.**

Proof:

$$\chi_{A\cup B}(x) = 1 \Leftrightarrow x \in A \cup B$$
$$\Leftrightarrow x \in A\ or\ x \in B$$
$$\Leftrightarrow \chi_A(x) = 1\ or\ \chi_B(x) = 1$$
$$\Leftrightarrow \chi_A(x) + \chi_B(x) - \chi_A(x).\chi_B(x) = 1$$
$$\Leftrightarrow \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) = 1 \ldots (1)$$
$$\chi_{A\cup B}(x) = 0 \Leftrightarrow x \notin A \cup B$$
$$\Leftrightarrow x \notin A\ and\ x \notin B$$

$$\Leftrightarrow \chi_A(x) = 0 \; and \; \chi_B(x) = 0$$
$$\Leftrightarrow \chi_A(x) + \chi_B(x) - \chi_A(x).\chi_B(x) = 0$$
$$\Leftrightarrow \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) = 0 \ldots (2)$$

$From\ (1)\ and\ (2)\ we\ get$

$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

$\therefore |A \cup B| = |A| + |B| - |A \cap B|.$

ii)Using Characteristic function , prove that $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$.

Proof:

$\chi_{\overline{A\cup B}}(x) = 1 \quad \Leftrightarrow x \in (\overline{A \cup B})$

$\Leftrightarrow x \notin A \cup B$

$\Leftrightarrow x \notin A \; and \; x \notin B$

$\Leftrightarrow x \in \bar{A} \; and \; x \in \bar{B}$

$\Leftrightarrow x \in \bar{A} \cap \bar{B}$

$\Leftrightarrow \chi_{\bar{A}\cap\bar{B}}(x) = 1 \ldots (1)$

$\chi_{\overline{A\cup B}}(x) = 0 \quad \Leftrightarrow x \notin (\overline{A \cup B})$

$\Leftrightarrow x \in A \cup B$

$\Leftrightarrow x \in A \; or \; x \in B$

$\Leftrightarrow x \notin \bar{A} \; or \; x \notin \bar{B}$

$\Leftrightarrow x \notin \bar{A} \cap \bar{B}$

$\Leftrightarrow \chi_{\bar{A}\cap\bar{B}}(x) = 0 \ldots (2)$

$From\ (1)\ and\ (2)\ we\ get$

$\chi_{\overline{A\cup B}}(x) = \chi_{\bar{A}\cap\bar{B}}(x), \qquad \forall x \in U$

$\therefore (\overline{A \cup B}) = \bar{A} \cap \bar{B}$