UNIT – 4
TAYLOR SERIES METHOD

The Taylor series algorithm is

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \cdots$$

$$(\boldsymbol{Or})\;\; y(x) = y_0 + \frac{h}{1\,!}\, y_0' + \frac{h^2}{2\,!}\, y_0'' + \frac{h^3}{3\,!}\, y_0''' + \frac{h^4}{4\,!}\, y_0^{iv} + \cdots \quad Where\;\; h = x_1 - x_0$$

Example. 1:

Using Taylor series method Find the value y at $x = 0.1$ if $\frac{dy}{dx} = x^2 y - 1,\;\; y(0\,) = 1.$

Solution : $Given \quad \frac{dy}{dx} = x^2 y - 1 \quad \& \quad y(0) = 1 \;\Rightarrow\; x_0 = 0,\; y_0 = 1 \qquad Since \quad [y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \cdots \; \ldots..(1)$$

$y' = x^2\, y - 1$	$y_0' = x_0^2\, y_0 - 1 = 0\,(1) - 1$	$y_0' = -1$
$y'' = x^2\, y' + y\, 2x - 0$	$y_0'' = x_0^2\, y_0' + 2\, y_0\, x_0 = 0(-1) + 2\,(1)(0)$	$y_0'' = 0$
$y''' = x^2\, y'' + y' 2\, x + 2y + 2x\, y'$ $y''' = x^2\, y'' + 4\, y'\, x + 2y$	$y_0''' = x_0^2\, y_0'' + 4\, y_0'\, x_0 + 2y_0$ $= 0(0) + 4(-1)(0) + 2(1)$	$y_0''' = 2$
$y^{iv} = x^2\, y''' + 2x\, y'' + 4\, y' + 4y'' x + 2y'$ $y^{iv} = x^2\, y''' + 6x\, y'' + 6\, y'$	$y_0^{iv} = x_0^2\, y_0''' + 6x_0\, y_0'' + 6\, y_0'$ $= 0(0) + 6(0)\,(0) + 6\,(-1)$	$y_0^{iv} = -6$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$$= 1 + \frac{(x-0)}{1}\,(-1) + \frac{(x-0)^2}{2}\,(0) + \frac{(x-0)^3}{6}\,(2) + \frac{(x-0)^4}{24}\,(-6) + \cdots$$

$$y(x) = 1 + x\,(-1) + \frac{x^2}{2}\,(0) + \frac{x^3}{6}\,(2) + \frac{x^4}{24}\,(-6)$$

To find y at $x = 0.1$

$$\therefore \quad y(0.1) = 1 + (0.1)\,(-1) + \frac{(0.1)^2}{2}\,(0) + \frac{(0.1)^3}{6}\,(2) + \frac{(0.1)^4}{24}\,(-6)$$

$$= 1 - 0.1 + 0 + 0.000333333 - 0.000025$$

$\therefore \quad \boldsymbol{y(0.1) \;= 0.900305}$

Example. 2:

Solve $y' = x + y\,,\ \ y(0) = 1$ by Taylor series method. Find the value $y\ \ at\ \ x = 0.1\ \ \&\ \ 0.2$.

Solution :

$$Given\quad y' = \frac{dy}{dx} = x + y \quad \& \quad y(0) = 1 \ \Rightarrow\ x_0 = 0, \qquad y_0 = 1 \qquad Since\quad [y(x_0) = y_0]$$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad .....(1)$$

$y' = x + y$	$y_0' = x_0 + y_0 = 0 + 1$	$y_0' = 1$
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 1 + 1$	$y_0'' = 2$
$y''' = 0 + y''$	$y_0''' = y_0'' = 2$	$y_0''' = 2$
$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$	$y_0^{iv} = 2$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad .....(1)$$

$$= 1 + \frac{(x - 0)}{1}\,(1) + \frac{(x - 0)^2}{2}\,(2) + \frac{(x - 0)^3}{6}\,(2) + \frac{(x - 0)^4}{24}\,(2) + \cdots$$

$$y(x) = 1 + x + \frac{x^2}{2}\,(2) + \frac{x^3}{6}\,(2) + \frac{x^4}{24}\,(2)$$

To find $y\ \ at\ \ x = 0.1$

$$\therefore\quad y(0.1) = 1 + (0.1) + \frac{(0.1)^2}{2}\,(2) + \frac{(0.1)^3}{6}\,(2) + \frac{(0.1)^4}{24}\,(2)$$

$$= 1 + 0.1 + 0.01 + 0.000333333 + 0.0000083333$$

$$\therefore\quad \mathbf{y(0.1)\ = 1.11034}$$

To find $y\ \ at\ \ x = 0.2$

$$\therefore\quad y(0.2) = 1 + (0.2) + \frac{(0.2)^2}{2}\,(2) + \frac{(0.2)^3}{6}\,(2) + \frac{(0.2)^4}{24}\,(2)$$

$$= 1 + 0.2 + 0.04 + 0.0026667 + 0.00013333$$

$$\therefore\quad \mathbf{y(0.1)\ = 1.2428000}$$

Example. 3: Solve $\frac{dy}{dx} = y^2 + x^2\ \ with\ \ y(0) = 1$. Use Taylor's method $at\ \ x = 0.2\ \ and\ \ 0.4$.

Solution : $Given \quad \frac{dy}{dx} = y^2 + x^2 \quad \& \quad y(0) = 1 \implies x_0 = 0,\ y_0 = 1 \quad Since \quad [y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$y' = y^2 + x^2$	$y_0' = y_0^2 + x_0^2 = 1 + 0$	$y_0' = 1$
$y'' = 2y\, y' + 2x$	$y_0'' = 2y_0\, y_0' + 2x_0 = 2(1)(1) + 2(0)$	$y_0'' = 2$
$y''' = 2y\, y'' + 2y'y' + 2$ $y''' = 2y\, y'' + 2(y')^2 + 2$	$y_0''' = 2y_0\, y_0'' + 2(y_0')^2 + 2$ $= 2(1)2 + 2(1) + 2$	$y_0''' = 8$
$y'^{v} = 2y\, y''' + 2y'y'' + 4y'\, y''$ $y'^{v} = 2y\, y''' + 6y'y''$	$y_0'^{v} = 2y_0\, y_0''' + 6y_0'y_0''$ $= 2(1)8 + 6(1)2$	$y_0'^{v} = 28$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$$= 1 + \frac{(x - 0)}{1}\,(1) + \frac{(x - 0)^2}{2}\,(2) + \frac{(x - 0)^3}{6}\,(8) + \frac{(x - 0)^4}{24}\,(28) + \cdots$$

$$y(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}\,(8) + \frac{x^4}{24}\,(28)$$

To find y at $x = 0.2$

$$\therefore \quad y(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{6}\,(8) + \frac{(0.2)^4}{24}\,(28)$$

$$= 1 + 0.2 + 0.02 + 0.010667 + 0.00186667$$

$$\therefore \quad \boldsymbol{y(0.2) = 1.23253}$$

To find y at $x = 0.4$

$$\therefore \quad y(0.4) = 1 + 0.4 + \frac{(0.4)^2}{2} + \frac{(0.4)^3}{6}\,(8) + \frac{(0.4)^4}{24}\,(28)$$

$$= 1 + 0.4 + 0.08 + 0.085333 + 0.0298667$$

$$\therefore \quad \boldsymbol{y(0.4) = 1.5952}$$

Example. 4: Using Taylor series method with the first five terms in the expansion find $y(0.1)$ correct to three decimal places, given that $\frac{dy}{dx} = e^x - y^2,\ y(0) = 1.$

Solution : $\quad Given \quad \frac{dy}{dx} = e^x - y^2 \quad \& \quad y(0) = 1 \implies x_0 = 0, \; y_0 = 1 \qquad Since \quad [y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \frac{(x-x_0)^5}{5\,!}\, y_0^{v} \ldots \quad \ldots\ldots(1)$$

$y' = e^x - y^2$	$y_0' = e^{x_0} - y_0^2 = e^0 - 1 = 1 - 1$	$y_0' = 0$
$y'' = e^x - 2y\, y'$	$y_0'' = e^{x_0} - 2y_0\, y_0' = 1 - 2(1)(0) = 1$	$y_0'' = 1$
$y''' = e^x - 2y\, y'' - 2y'y'$ $y''' = e^x - 2y\, y'' - 2(y')^2$	$y_0''' = e^{x_0} - 2y_0\, y_0'' - 2(y_0')^2$ $= 1 - 2(1)(1) - 2(0)$	$y_0''' = -1$
$y^{iv} = e^x - 2y\, y''' - 2y'y'' - 4y'\, y''$ $y^{iv} = e^x - 2y\, y''' - 6y'y''$	$y_0^{iv} = e^{x_0} - 2y_0\, y_0''' - 6y_0'y_0''$ $= 1 - 2(1)(-1) - 6(0)(1)$	$y_0^{iv} = 3$
$y^v = e^x - 2y\, y^{iv} - 2y'y''' - 6y'\, y''' - 6y''y''$ $y^v = e^x - 2y\, y^{iv} - 8y'y''' - 6(y'')^2$	$y^v = e^{x_0} - 2y_0\, y_0^{iv} - 8y_0'y_0''' - 6(y_0'')^2$ $= 1 - 2(1)(3) - 8(0)(-1) - 6(1)^2$	$y^v = -11$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \frac{(x-x_0)^5}{5\,!}\, y_0^{v} \ldots \quad \ldots\ldots(1)$$

$$= 1 + \frac{(x-0)}{1}\,(0) + \frac{(x-0)^2}{2}\,(1) + \frac{(x-0)^3}{6}\,(-1) + \frac{(x-0)^4}{24}\,(3) + \frac{(x-0)^5}{120}\,(-11) \ldots$$

$$y(x) = 1 + x\,(0) + \frac{x^2}{2}\,(1) + \frac{x^3}{6}\,(-1) + \frac{x^4}{24}\,(3) + \frac{x^5}{120}\,(-11) \ldots$$

To find $y(0.1)$: $[y \;\; at \;\; x = 0.1]$

$$\therefore \quad y(0.1) = 1 + 0.1\,(0) + \frac{(0.1)^2}{2}\,(1) + \frac{(0.1)^3}{6}\,(-1) + \frac{(0.1)^4}{24}\,(3) + \frac{(0.1)^5}{120}\,(-11)$$

$$= 1 + 0 + 0.005 - 0.00016667 + 0.0000125 - 0.00000091667$$

$\therefore \quad \boldsymbol{y(0.2) = 1.004844}$

Example. 5: Using Taylor series method Find $y(0.2)$ & $y(0.4)$ correct to four decimal places given $\frac{dy}{dx} = 1 - 2\,x\,y, \;\; y(0) = 0.$

Solution : $\quad Given \quad \frac{dy}{dx} = y' = 1 - 2\,x\,y \quad \& \quad y(0) = 0 \implies x_0 = 0, \; y_0 = 0 \qquad Since \quad [y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\,y_0' + \frac{(x-x_0)^2}{2\,!}\,y_0'' + \frac{(x-x_0)^3}{3\,!}\,y_0''' + \frac{(x-x_0)^4}{4\,!}\,y_0^{iv} + \frac{(x-x_0)^5}{5\,!}\,y_0^{v} + \cdots \quad \ldots..(1)$$

$y' = 1 - 2\,x\,y$	$y_0' = 1 - 2\,x_0\,y_0 = 1 - 2(0)(0)$	$y_0' = 1$
$y'' = 0 - 2\,x\,y' - 2y$	$y_0'' = -2\,x_0\,y_0' - 2y_0$ $= -2(0)(1) - 2(0)$	$y_0'' = 0$
$y''' = -2\,x\,y'' - 2y' - 2y'$ $y''' = -2\,x\,y'' - 4\,y'$	$y_0''' = -2\,x_0\,y_0'' - 4\,y_0'$ $= -2(0)(0) - 4(1)$	$y_0''' = -4$
$y'^{v} = -2\,x\,y''' - 2\,y'' - 4\,y''$ $y'^{v} = -2\,x\,y''' - 6\,y''$	$y_0'^{v} = -2\,x_0\,y_0''' - 6\,y_0''$ $= -2(0)(-4) - 6(0)$	$y_0'^{v} = 0$
$y^{v} = -2\,x\,y'^{v} - 2\,y''' - 6\,y'''$ $y'^{v} = -2\,x\,y'^{v} - 8\,y'''$	$y_0'^{v} = -2\,x_0\,y_0'^{v} - 8\,y_0'''$ $= -2(0)(0) - 8(-4)$	$y_0'^{v} = 32$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\,y_0' + \frac{(x-x_0)^2}{2\,!}\,y_0'' + \frac{(x-x_0)^3}{3\,!}\,y_0''' + \frac{(x-x_0)^4}{4\,!}\,y_0^{iv} + \frac{(x-x_0)^5}{5\,!}\,y_0^{v} + \cdots \quad \ldots..(1)$$

$$= 0 + \frac{(x-0)}{1}\,(1) + \frac{(x-0)^2}{2}\,(0) + \frac{(x-0)^3}{6}\,(-4) + \frac{(x-0)^4}{24}\,(0) + \frac{(x-0)^5}{120}\,(32) \ldots$$

$$y(x) = x + \frac{x^3}{6}\,(-4) + \frac{x^5}{120}\,(32)$$

To find $y(0.2)$:

$$\therefore \quad y(0.2) = 0.2 + \frac{(0.2)^3}{6}\,(-4) + \frac{(0.2)^5}{120}\,(32)$$

$$= 0.2 - 0.005333 + 0.00008533$$

$$\therefore \quad \boldsymbol{y(0.1) = 0.194752}$$

To find $y(0.4)$:

$$\therefore \quad y(0.4) = 0.4 + \frac{(0.4)^3}{6}\,(-4) + \frac{(0.4)^5}{120}\,(32)$$

$$= 0.4 - 0.0426667 + 0.002730667$$

$$\therefore \quad \boldsymbol{y(0.1) = 0.360063}$$

Example. 6: Using Taylor series method Find y at $x = 0.1$ correct to four decimal places given $\frac{dy}{dx} = x^2 - y,\ y(0) = 1$. Take $h = 0.1$

Solution : *Given* $\frac{dy}{dx} = y' = x^2 - y$ & $y(0) = 1 \Rightarrow x_0 = 0,\ y_0 = 1$ *Since* $[y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$y' = x^2 - y$	$y_0' = x_0^2 - y_0 = 0 - 1$	$y_0' = -1$
$y'' = 2x - y'$	$y_0'' = 2\,x_0 - y_0' = 2(0) - (-1)$	$y_0'' = +1$
$y''' = 2 - y''$	$y_0''' = 2 - y_0'' = 2 - 1$	$y_0''' = 1$
$y^{iv} = 0 - y'''$	$y_0^{iv} = -\,y_0''' = -1$	$y_0^{iv} = -1$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$$= 1 + \frac{(x - 0)}{1}\,(-1) + \frac{(x - 0)^2}{2}\,(1) + \frac{(x - 0)^3}{6}\,(1) + \frac{(x - 0)^4}{24}\,(-1)$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}(-1)$$

To find $\boldsymbol{y(0.1)}$ **:** $[\boldsymbol{y\ \ at\ \ x = 0.1}]$

$$\therefore\quad y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{6} + \frac{(0.1)^4}{24}(-1)$$

$$= 1 - 0.1 + 0.005 + 0.00016667 - 0.00000416667$$

$$\therefore\quad \boldsymbol{y(0.1) = 0.90516}$$

Example. 7: **Using Taylor series method, Find** $\boldsymbol{y(1.1)}$ **given** $\boldsymbol{y' = x + y,\ \ y(1) = 0.}$

Solution : *Given* $\frac{dy}{dx} = y' = x + y$ & $y(1) = 0 \Rightarrow x_0 = 1,\ y_0 = 0$ *Since* $[y(x_0) = y_0]$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\, y_0' + \frac{(x - x_0)^2}{2\,!}\, y_0'' + \frac{(x - x_0)^3}{3\,!}\, y_0''' + \frac{(x - x_0)^4}{4\,!}\, y_0^{iv} + \cdots \quad \ldots..(1)$$

$y' = x + y$	$y_0' = x_0 + y_0 = 1 + 0$	$y_0' = 1$
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 1 + 1$	$y_0'' = 2$
$y''' = y''$	$y_0''' = y_0'' = 2$	$y_0''' = 2$

$y^{iv} = y'''$	$y_0^{iv} = y_0''' = 2$	$y_0^{iv} = 2$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x-x_0)}{1\,!}\, y_0' + \frac{(x-x_0)^2}{2\,!}\, y_0'' + \frac{(x-x_0)^3}{3\,!}\, y_0''' + \frac{(x-x_0)^4}{4\,!}\, y_0^{iv} + \cdots \qquad \ldots..(1)$$

$$= 0 + \frac{(x-1)}{1}\,(1) + \frac{(x-1)^2}{2}\,(2) + \frac{(x-1)^3}{6}\,(2) + \frac{(x-1)^4}{24}\,(2)$$

$$y(x) = (x-1) + (x-1)^2 + \frac{(x-1)^3}{3} + \frac{(x-1)^4}{12}$$

To find $y(1.1)$: $[y \ at \ x = 1.1]$

$$\therefore \quad y(1.1) = (1.1-1) + (1.1-1)^2 + \frac{(1.1-1)^3}{3} + \frac{(1.1-1)^4}{12}$$

$$= (0.1) + (0.1)^2 + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{12}$$

$$= 0.1 + 0.01 + 0.0003333 + 0.0000083333$$

$\therefore \quad \boldsymbol{y(0.1) = 0.11034}$

EULER'S METHOD & MODIFIED EULER'S METHOD

The Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\,[f(x_n, y_n)] \ , \ n = 0,1,2,\ldots.. \qquad \ldots\ldots\ldots(1)}$$

Example . 1 : Given $\boldsymbol{y' = -y}$ and $\boldsymbol{y(0) = 1}$, determine the values of $y \ at \ x = (0.01)\,(0.01)\,(0.04)$ by Euler's method.

Solution : $Given \ \ y' = -y \ \ and \ \ y(0) = 1 \ \Rightarrow \ x_0 = 0, \ y_0 = 1 \qquad [Since \ \ y(x_0) = y_0]$

$\therefore \ f(x,y) = -y$

To find h : $Since \ y(0) = 1 \ \Rightarrow \ y(x_0) = y_0$

$We\,need\,to\,find \ \ y \ at \ x = (0.01)\,(0.01)\,(0.04)$

$\Rightarrow \ y(x_1) = ?, \ [y(0.01) = ?] \ \ \& \ \ y(x_2) = ? \ [y(0.02) = ?] \ \ldots \ \Rightarrow \ x_1 = 0.01, \quad x_2 = 0.02 \ \ldots$

$\therefore \ h = x_1 - x_0 = 0.01 - 0.0 = 0.01 \quad (or) \quad h = x_2 - x_1 = 0.02 - 0.01 = 0.01 \quad [Difference]$

The Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\,[f(x_n, y_n)] \ , \ n = 0,1,2,\ldots.. \qquad \ldots\ldots\ldots(1)}$$

To find $y(0.01)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + \ h\,[f(x_0, y_0)]$$

$We\ have\quad x_0 = 0,\ \ y_0 = 1\ ,\ \ h = 0.01\ \ \&\ \ f(x,y) = -y$

$\therefore\quad y_1(0 + 0.01) = 1 + (0.01)\,[f(0,1\,)]$

$$y_1(0.01) = 1 + (0.01)\,[-1] = 1 - 0.01$$

$\boldsymbol{y_1(0.01) = 0.99}\qquad [\boldsymbol{y(x_1) = y_1}]\quad \Rightarrow\ \ x_1 = \mathbf{0.01}\ \ \&\ \boldsymbol{y_1 = 0.99}$

To find $\boldsymbol{y(0.02)}$:

Put $\boldsymbol{n = 1}$, equation $(\mathbf{1})$ becomes

$$y_2(x_1 + h) = y_1 + \ h\,[f(x_1, y_1)]$$

$We\ have\quad x_1 = 0.01\ \ \&\ y_1 = 0.99\,,\ \ h = 0.01\ \ \&\ \ f(x,y) = -y$

$\therefore\quad y_2(0.01 + 0.01) = 0.99 + (0.01)\,[f(0.01, 0.99\,)]$

$$y_2(0.02) = 0.99 + (0.01)\,[-0.99] = 0.99 - 0.0099$$

$y_2(\mathbf{0.02}) = \mathbf{0.9801}\qquad [\boldsymbol{y(x_2) = y_2}]\quad \Rightarrow\ \ x_2 = \mathbf{0.02}\ \ \&\ \boldsymbol{y_2} = \mathbf{0.9801}$

To find $\boldsymbol{y(0.03)}$:

Put $\boldsymbol{n = 2}$, equation $(\mathbf{1})$ becomes

$$y_3(x_2 + h) = y_2 + \ h\,[f(x_2, y_2)]$$

$We\ have\quad x_2 = 0.02\ \ \&\ y_2 = 0.9801\,,\ \ h = 0.01\ \ \&\ \ f(x,y) = -y$

$\therefore\quad y_3(0.02 + 0.01) = 0.9801 + (0.01)\,[f(0.02, 0.9801\,)]$

$$y_3(0.02) = 0.9801 + (0.01)\,[-0.9801] = 0.9801 - 0.009801$$

$y_3(\mathbf{0.02}) = \mathbf{0.970299}\qquad [\boldsymbol{y(x_3) = y_3}]\quad \Rightarrow\ \ \boldsymbol{x_3 = 0.03}\ \ \&\ \boldsymbol{y_3 = 0.970299}$

To find $\boldsymbol{y(0.04)}$:

Put $\boldsymbol{n = 3}$, equation $(\mathbf{1})$ becomes

$$y_4(x_3 + h) = y_3 + \ h\,[f(x_3, y_3)]$$

$We\ have\quad x_3 = 0.03\ \ \&\ y_3 = 0.970299\,,\ \ h = 0.01\ \ \&\ \ f(x,y) = -y$

$\therefore\quad y_4(0.02 + 0.01) = 0.970299 + (0.01)\,[f(0.03, 0.970299\,)]$

$$y_4(0.02) = 0.970299 + (0.01)\,[-0.970299] = 0.970299 - 0.009702999$$

$y_4(\mathbf{0.02}) = \mathbf{0.96059}\qquad [\boldsymbol{y(x_4) = y_4}]\quad \Rightarrow\ \ \boldsymbol{x_4 = 0.03}\ \ \&\ \boldsymbol{y_4 = 0.96059}$

Example . 2 : Using Euler's method Solve numerically the equation

$y' = x + y \quad , \quad y(0) = 1 \quad for \;\; x = 0.0\,(0.2)\,(1.0).$

Solution : $Given \quad y' = x + y \quad and \;\; y(0) = 1 \quad \Rightarrow \; x_0 = 0, \;\; y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$

$\therefore \;\; f(x, y) = x + y$

To find h : $Since \;\; y(0) = 1 \quad \Rightarrow \quad y(x_0) = y_0$

$We\; need\; to\; find \;\; y \;\; at \;\; x = (0.0)\,(0.2)\,(1.0)$

$\Rightarrow \;\; y(x_1) = ?, \;\; [y(0.2) = ?] \quad \& \quad y(x_2) = ? \;\; [y(0.4) = ?] \;\; \ldots \quad \Rightarrow \; x_1 = 0.2, \qquad x_2 = 0.4 \;\; \ldots$

$\therefore \;\; h = x_1 - x_0 = 0.2 - 0.0 = 0.2 \qquad (or) \qquad h = x_2 - x_1 = 0.4 - 0.2 = 0.2 \quad [Difference]$

The Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + \; h\,[f(x_n, y_n)] \;\; , \;\; n = 0, 1, 2, \ldots.. \qquad \ldots\ldots\ldots(1)}$$

To find $y(0.2)$:

Put $n = 0$, equation (1) becomes

$y_1(x_0 + h) = y_0 + \; h\,[f(x_0, y_0)]$

$We\; have \quad x_0 = 0, \;\; y_0 = 1 \;\; , \;\; h = 0.2 \;\; \& \;\; f(x, y) = x + y$

$\therefore \quad y_1(0 + 0.2) = 1 + (0.2)\,[f(0, 1\,)]$

$y_1(0.2) = 1 + (0.2)\,[0 + 1] = 1 + 0.2$

$\boldsymbol{y_1(0.2) = 1.2} \qquad \boldsymbol{[y(x_1) = y_1]} \;\; \Rightarrow \;\; \boldsymbol{x_1 = 0.2 \;\; \& \;\; y_1 = 1.2}$

To find $y(0.4)$:

Put $\boldsymbol{n = 1}$, equation $(\boldsymbol{1})$ becomes

$y_2(x_1 + h) = y_1 + \; h\,[f(x_1, y_1)]$

$We\; have \quad x_1 = 0.2 \;\; \& \;\; y_1 = 1.2\,, \;\; h = 0.01 \;\; \& \;\; f(x, y) = x + y$

$\therefore \quad y_2(0.2 + 0.2) = 1.2 + (0.2)\,[f(0.2, \;\; 1.2\,)]$

$y_2(0.4) = 1.2 + (0.2)\,[0.2 + 1.2] = 1.2 + 0.28$

$y_2(\boldsymbol{0.4}) = \boldsymbol{1.48} \qquad [\boldsymbol{y(x_2) = y_2}] \quad \Rightarrow \;\; x_2 = \boldsymbol{0.4} \;\; \& \;\; \boldsymbol{y_2 = 1.48}$

To find $y(0.6)$:

Put $\boldsymbol{n = 2}$, equation $(\boldsymbol{1})$ becomes

$y_3(x_2 + h) = y_2 + \; h\,[f(x_2, y_2)]$

$We\ have\quad x_2 = 0.4\ \ \&\ y_2 = 1.48\,,\ \ h = 0.2\ \ \&\ \ f(x,y) = x + y$

$\therefore\quad y_3(0.4 + 0.2) = 1.48 + (0.01)\ [f(0.4\,,\ \ 1.48\,)]$

$$y_3(0.6) = 1.48 + (0.2)\ [1.48 + 0.4] = 1.48 + 0.176$$

$y_3(\mathbf{0.6}) = \mathbf{1.656}\qquad [\boldsymbol{y}(\boldsymbol{x_3}) = \boldsymbol{y_3}]\quad \Longrightarrow\ \boldsymbol{x_3} = \mathbf{0.6}\ \ \&\ \boldsymbol{y_3} = \mathbf{1.656}$

To find $y(0.8)$:

Put $n = 3$, equation (1) becomes

$$y_4(x_3 + h) = y_3 +\ h\ [f(x_3, y_3)]$$

$We\ have\quad x_3 = 0.6\ \ \&\ y_3 = 1.656\,,\ \ h = 0.2\ \ \&\ \ f(x,y) = x + y$

$\therefore\quad y_4(0.6 + 0.2) = 1.656 + (0.2)\ [f(0.6,\ \ 1.656\,)]$

$$y_4(0.8) = 1.656 + (0.2)\ [0.6 + 1.656] = 1.656 + 0.4512$$

$y_4(\mathbf{0.8}) = \mathbf{2.1072}\qquad [\boldsymbol{y}(\boldsymbol{x_4}) = \boldsymbol{y_4}]\quad \Longrightarrow\ \boldsymbol{x_4} = \mathbf{0.8}\ \ \&\ \boldsymbol{y_4} = \mathbf{0.96059}$

To find $y(1.0)$:

Put $n = 4$, equation (1) becomes

$$y_5(x_4 + h) = y_4 +\ h\ [f(x_4, y_4)]$$

$We\ have\quad x_4 = 0.8\ \ \&\ y_4 = 2.1072\,,\ \ h = 0.2\ \ \&\ \ f(x,y) = x + y$

$\therefore\quad y_5(0.8 + 0.2) = 2.1072 + (0.2)\ [f(0.8,\ \ 2.1072\,)]$

$$= 1.656 + (0.2)\ [0.6 + 1.656] = 1.656 + 0.4512$$

$y_5(\mathbf{1.0}) = \mathbf{2.1072}\qquad [\boldsymbol{y}(\boldsymbol{x_5}) = y_5]\quad \Longrightarrow\ \boldsymbol{x_4} = \mathbf{1.0}\ \ \&\ y_5 = \mathbf{2.1072}$

Example . 3 : Using Euler's find $y(0.3)\ \ of\ \ y(x)$ satisfies the initial value problem

$\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2\ ,\ y(0.2) = 1.1114$

Solution : Given $y' = f(x,y) = \frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2\quad and\ \ y(0.2) = 1.1114\ \Longrightarrow\ x_0 = 0.2,\ \ y_0 = 1.1114$

$$\therefore\ \ f(x,y) = \frac{1}{2}(1 + x^2)y^2$$

To find h :

$Since\ \ y(0.2) = 1.1114\qquad \Longrightarrow\quad y(x_0) = y_0$

$We\ need\ to\ find\quad y\ \ at\ \ x = 0.3$

$\Longrightarrow\quad y(x_1) = ?,\ \ [y(0.3) = ?]\qquad \Longrightarrow\ \ x_1 = 0.3,$

$\therefore\ h = x_1 - x_0 = 0.3 - 0.2 = 0.1 \qquad [Difference]$

The Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\,[f(x_n, y_n)]\ ,\ n = 0,1,2,\ldots..} \qquad \ldots\ldots\ldots(1)$$

To find $y(0.3)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\,[f(x_0, y_0)]$$

$We\ have\quad x_0 = 0.2,\ y_0 = 1.1114\ ,\ h = 0.1\ \ \&\ \ f(x,y) = \frac{1}{2}(1 + x^2)y^2$

$\therefore\quad y_1(0.2 + 0.1) = 1.1114 + (0.1)\,[f(0.2, 1.1114\,)]$

$$= 1 + (0.1)\left[\frac{1}{2}(1 + (0.2)^2)(1.1114)^2\right] = 1.1114 + 0.1\,[0.642309]$$

$\boldsymbol{y_1(0.3) = 1.17564} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow\ x_1 = \boldsymbol{0.3}\ \ \&\ \boldsymbol{y_1 = 1.17564}$

Example . 4 : Using Euler's method find the solution of the initial value problem

$\frac{dy}{dx} = \log(x + y)\ ,\ y(0) = 2\ \ at\ \ x = 0.2$ by assuming $h = 0.2$

Solution : Given

$y' = f(x,y) = \frac{dy}{dx} = \log(x + y) \qquad and\ \ y(0) = 0\ \Longrightarrow\ x_0 = 0,\ y_0 = 2 \qquad [Since\quad y(x_0) = y_0]$

$\therefore\ \ f(x,y) = \log(x + y)\,,\ \ h = 0.2$

The Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\,[f(x_n, y_n)]\ ,\ n = 0,1,2,\ldots..} \qquad \ldots\ldots\ldots(1)$$

To find $\boldsymbol{y\ \ at\ \ x = 0.2\quad [y(0.3)]}$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\,[f(x_0, y_0)]$$

$We\ have\quad x_0 = 0,\ \ y_0 = 2\ ,\ \ h = 0.2\ \ \&\ \ f(x,y) = \log(x + y)$

$\therefore\quad y_1(0 + 0.2) = 2 + (0.2)\,[f(0{,}2\,)]$

$$y_1(0.2) = 2 + (0.2)\,[\log(0 + 2)] = 2 + 0.2\,[\log 2]$$

$$= 2 + (0.2)\,[0.301029] = 2.060205$$

$\boldsymbol{y_1(0.2) = 2.060205} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow\ x_1 = \boldsymbol{0.2}\ \ \&\ \boldsymbol{y_1 = 2.060205}$

MODIFIED EULER'S METHOD

$$y_{n+1}(x_n + h) = y_n + h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right) \quad , \ n = 0,1,2, \ldots\ldots \qquad \ldots\ldots\ldots(1)$$

Example . 5 : By Modified Euler's method, compute $y(0.1)$ $with$ $h = 0.1$ $from$ $\frac{dy}{dx} = y - \frac{2x}{y}$, $y(0) = 1$

Solution : Given

$$y' = f(x,y) = \frac{dy}{dx} = y - \frac{2x}{y} \quad and \quad y(0) = 1 \ \Rightarrow \ x_0 = 0,\ y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$$

$$\therefore \ f(x,y) = y - \frac{2x}{y} \ , \ h = 0.1$$

The Modified Euler's formula is

$$y_{n+1}(x_n + h) = y_n + h\, f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2} f(x_n, y_n)\right) \quad , \ n = 0,1,2, \ldots\ldots \qquad \ldots\ldots\ldots(1)$$

To find y at $x = 0.1$ $[y(0.1)]$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\, f\left(x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\right)$$

$We\ have$ $x_0 = 0,\ y_0 = 1$, $h = 0.1$ & $f(x,y) = y - \frac{2x}{y}$

$$\therefore \quad y_1(0 + 0.1) = 1 + (0.1)\ f\left(0 + \frac{0.1}{2},\ 1 + \frac{0.1}{2} f(0,1)\right)$$

$$y_1(0.1) = 1 + (0.1)\, f\left(0.05,\ 1 + 0.05\left[1 - \frac{2(0)}{1}\right]\right)$$

$$= 1 + (0.1)\, f(0.05,\ 1 + 0.05\,[1])$$

$$= 1 + (0.1)\, f(0.05,\ 1.05)$$

$$= 1 + (0.1)\left[1.05 - \frac{2(0.05)}{1.05}\right]$$

$$= 1 + (0.1)\,[1.05 - 0.0952]$$

$$= 1 + 0.09548$$

$$y_1(0.1) = 1.09548$$

$$\boldsymbol{y_1(0.1) = 1.09548} \qquad [y(x_1) = y_1] \quad \Rightarrow \ x_1 = \mathbf{0.1} \ \ \& \ \boldsymbol{y_1 = 1.09548}$$

Example . 6 : Using Modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Solution : Given

$$y' = f(x,y) = \frac{dy}{dx} = x^2 + y^2 \quad and \quad y(0) = 1 \quad \Longrightarrow \quad x_0 = 0, \; y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$$

To find h :

$$Since \; y(0) = 1 \qquad \Longrightarrow \quad y(x_0) = y_0$$

$$Also\ we\ need\ to\ find \quad y(0.1) \quad \Longrightarrow \quad y(x_1) = ?, \;\; [y(0.1) = ?] \qquad \Longrightarrow \quad x_1 = 0.1,$$

$$\therefore \; h = x_1 - x_0 = 0.1 - 0 = 0.1 \qquad [Difference]$$

The Modified Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\, f\left(x_n + \frac{h}{2},\; y_n + \frac{h}{2}\, f(x_n, y_n)\right)} \;, \; \boldsymbol{n = 0, 1, 2, \ldots\ldots} \qquad \ldots\ldots\ldots(\mathbf{1})$$

To find $y(0.1)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\, f\left(x_0 + \frac{h}{2},\; y_0 + \frac{h}{2}\, f(x_0, y_0)\right)$$

$$We\ have \quad x_0 = 0, \;\; y_0 = 1 \;, \;\; h = 0.1 \;\; \& \;\; f(x,y) = x^2 + y^2$$

$$\therefore \quad y_1(0 + 0.1) = 1 + (0.1)\; f\left(0 + \frac{0.1}{2},\; 1 + \frac{0.1}{2}\, f(0,1)\right)$$

$$\begin{aligned} y_1(0.1) &= 1 + (0.1)\, f(0.05,\; 1 + 0.05\,[0^2 + 1^2]) \\ &= 1 + (0.1)\, f(0.05,\; 1 + 0.05\,[1]) \\ &= 1 + (0.1)\, f(0.05,\; 1.05) \\ &= 1 + (0.1)\,[(0.05)^2 + (1.05)^2] \\ &= 1 + (0.1)\,[1.105] \\ &= 1 + 0.1105 \end{aligned}$$

$$y_1(0.1) = 1.1105$$

$$\boldsymbol{y_1(0.1) = 1.1105} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow \quad x_1 = \mathbf{0.1} \;\; \& \;\; \boldsymbol{y_1 = 1.1105}$$

Example . 7 :

Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1, \; y(0) = 0.5$. Using Modified Euler's method, find $y(0.2)$.

Solution : Given $y' = f(x,y) = \frac{dy}{dx} = y - x^2 + 1 \quad and \quad y(0) = 0.5 \quad \Longrightarrow \quad x_0 = 0, \; y_0 = 0.5$

To find h :

$Since\ \ y(0) = 0.5 \qquad \Longrightarrow \quad y(x_0) = y_0$

$Also\ we\ need\ to\ find\ \ y(0.2) \quad \Longrightarrow \quad y(x_1) = ?,\ \ [y(0.2) = ?] \qquad \Longrightarrow \quad x_1 = 0.2,$

$\therefore\ \ \boldsymbol{h = x_1 - x_0 = 0.2 - 0 = 0.2} \qquad [\boldsymbol{Difference}]$

The Modified Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\, f\left(x_n + \frac{h}{2},\ \ y_n + \frac{h}{2}\, f(x_n, y_n)\right)\ \ ,\ \ n = 0, 1, 2, \ldots.. \qquad \ldots\ldots\ldots(1)}$$

To find $y(0.2)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\, f\left(x_0 + \frac{h}{2},\ \ y_0 + \frac{h}{2}\, f(x_0, y_0)\right)$$

$We\ have\ \ \ x_0 = 0,\ \ y_0 = 0.5\ \ ,\ \ h = 0.1\ \ \&\ \ f(x,y) = y - x^2 + 1$

$$\therefore \quad y_1(0 + 0.2) = 0.5 + (0.2)\ f\left(0 + \frac{0.2}{2},\ \ 0.5 + \frac{0.2}{2}\ f(0, 0.5)\right)$$

$$y_1(0.2)\ = 1 + (0.2)\, f(0.1,\ \ 1 + 0.1\, [0.5 - 0^2 + 1])$$

$$= 0.5 + (0.2)\, f(0.1,\ \ 0.5 + 0.1\, [1.5])$$

$$= 0.5 + (0.2)\, f(0.1,\ \ 0.65)$$

$$= 0.5 + (0.2)\, [0.65 - (0.1)^2 + 1]$$

$$= 0.5 + (0.2)\, [1.64]$$

$$= 0.5 + 0.328$$

$$y_1(0.2)\ = 0.828$$

$\boldsymbol{y_1(0.2) = 0.828} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow \quad x_1 = \boldsymbol{0.2\ \ \&\ \ y_1 = 0.828}$

Example . 8 : Solve $y' = 1 - y$, $y(0) = 0$ by using Modified Euler's method.

Solution : Given

$y' = f(x,y) = \frac{dy}{dx} = 1 - y \qquad and\ \ y(0) = 0 \quad \Longrightarrow \quad x_0 = 0,\ \ y_0 = 0 \qquad [Since \quad y(x_0) = y_0]$

To find h :

$Assume\ h = 0.1$

The Modified Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + h\, f\left(x_n + \frac{h}{2},\ \ y_n + \frac{h}{2}\, f(x_n, y_n)\right)\ \ ,\ \ n = 0, 1, 2, \ldots\ldots} \qquad \ldots\ldots\ldots(1)$$

To find $y(0.1)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + h\, f\left(x_0 + \frac{h}{2},\ \ y_0 + \frac{h}{2}\, f(x_0, y_0)\right)$$

$We\ have \quad x_0 = 0,\ \ y_0 = 0\ \ ,\ \ h = 0.1\ \ \&\ \ f(x, y) = 1 - y$

$$\therefore \quad y_1(0 + 0.1) = 0 + (0.1)\ f\left(0 + \frac{0.1}{2},\ \ 0 + \frac{0.1}{2}\, f(0, 0)\right)$$

$$y_1(0.1) = 0 + (0.1)\, f(0.05,\ \ 0 + 0.05\,[1 - 0])$$

$$= (0.1)\, f(0.05,\ \ 0 + 0.05\,[1])$$

$$= (0.1)\, f(0.05,\ \ 0.05)$$

$$= (0.1)\,[1 - 0.051]$$

$$= (0.1)\,[0.95]$$

$$= 0.095$$

$$y_1(0.1) = 0.095$$

$\boldsymbol{y_1(0.1) = 0.095}$ $\quad [y(x_1) = y_1] \quad \Rightarrow \quad x_1 = \mathbf{0.1}\ \ \&\ \ \boldsymbol{y_1 = 0.095}$

To find $y(0.2)$:

Put $n = 1$, equation (1) becomes

$$y_2(x_1 + h) = y_1 + h\, f\left(x_1 + \frac{h}{2},\ \ y_1 + \frac{h}{2}\, f(x_1, y_1)\right)$$

$We\ have \quad x_1 = 0.1,\ \ y_1 = 0.095\ \ ,\ \ h = 0.1\ \ \&\ \ f(x, y) = 1 - y$

$$\therefore \quad y_2(0.1 + 0.1) = 0.095 + (0.1)\ f\left(0.1 + \frac{0.1}{2},\ \ 0.095 + \frac{0.1}{2}\, f(0.1, 0.095)\right)$$

$$y_2(0.2) = 0.095 + (0.1)\, f(0.15,\ \ 0.095 + 0.05\,[1 - 0.095])$$

$$= 0.095 + (0.1)\, f(0.15,\ \ 0.095 + 0.05\,[0.905])$$

$$= 0.095 + (0.1)\, f(0.15,\ \ 0.14025)$$

$$= 0.095 + (0.1)\,[1 - 0.14025]$$

$= 0.095 + (0.1)\ [0.85975]$

$y_2(0.2)\ = 0.18098$

$\boldsymbol{y_2(0.2) = 0.18098} \qquad [\boldsymbol{y(x_2) = y_2}] \quad \Longrightarrow \ x_2 = \boldsymbol{0.2}\ \&\ \boldsymbol{y_2 = 0.18098}$

To find $\boldsymbol{y(0.3)}$:

Put $n = 2$, equation (1) becomes

$$y_3(x_2 + h) = y_2 +\ h\, f\left(x_2 + \frac{h}{2},\ \ y_2 + \frac{h}{2}\, f(x_2, y_2)\right)$$

$We\ have\quad x_2 = 0.2,\ \ y_2 = 0.180985\ ,\ \ h = 0.1\ \ \&\ \ f(x, y) = 1 - y$

$$\therefore \quad y_3(0.2 + 0.1) = 0.18098 + (0.1)\ f\left(0.2 + \frac{0.1}{2},\ \ 0.18098 + \frac{0.1}{2}\, f(0.2, 0.18098)\right)$$

$y_3(0.3)\ = 0.18098 + (0.1)\ \ f(0.25,\ \ 0.18098 + 0.05\ [1 - 0.18098])$

$=\ 0.18098 + (0.1)\ \ f(0.25,\ \ 0.18098 + 0.040951)$

$=\ 0.18098 + (0.1)\ \ f(0.25,\ \ 0.221931)$

$= 0.18098 + (0.1)\ \ [1 - 0.221931]$

$= 0.18098 + (0.1)\ \ [0.778069]$

$=\ 0.18098 + 0.0778069$

$y_3(0.3)\ = 0.2587869$

$\boldsymbol{y_3(0.3) = 0.2587869} \qquad [\boldsymbol{y(x_3) = y_3}] \quad \Longrightarrow \ \boldsymbol{x_3 = 0.32\ \&\ y_3 = 0.25878698}$

IMPROVED EULE'S METHOD

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + \left(\frac{h}{2}\right)\left[f(x_n, y_n) + f\big(x_n + h,\ \ y_n + h\, f(x_n, y_n)\big)\right]\ \ ,\ \ n = 0, 1, 2, \ldots.. \qquad \ldots\ldots\ldots(1)}$$

Example . 8 :

Find $\ y\ at\ x = 0.1,\ 0.2\ \ \&\ \ 0.3\ \ \ given\ \ \ y' = 1 - y\ ,\ y(0) = 0$ by using Improved Euler's method.

Solution : Given

$y' = f(x, y) = \frac{dy}{dx} = 1 - y \qquad and\ \ y(0) = 0\ \ \Longrightarrow\ x_0 = 0,\ \ y_0 = 0 \qquad [Since\quad y(x_0) = y_0]$

To find $\boldsymbol{h}$:

$Since\ \ y(0) = 0 \qquad \Longrightarrow \quad y(x_0) = y_0$

$Also\ we\ need\ to\ find\ \ \ y\ \ at\ \ x = 0.1, 0.2\ \&\ 0.3$

$\Rightarrow \quad y(x_1) = ?, \ [y(0.1) = ?] \quad \& \quad y(x_2) = ? \ [y(0.2) = ?] \ \ldots \quad \Rightarrow x_1 = 0.1, \quad x_2 = 0.2 \ \ldots$

$\therefore \ h = x_1 - x_0 = 0.1 - 0.0 = 0.1 \quad (or) \quad h = x_2 - x_1 = 0.2 - 0.1 = 0.1 \quad [Difference]$

$\therefore \ h = 0.1$

The Improved Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + \left(\frac{h}{2}\right) \left[f(x_n, y_n) + f\big(x_n + h, \ \ y_n + h\, f(x_n, y_n)\big)\right] \ , \ n = 0, 1, 2, \ldots.} \qquad \ldots\ldots\ldots(1)$$

To find $\boldsymbol{y(0.1)}$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + \left(\frac{h}{2}\right) \left[f(x_0, y_0) + f\big(x_0 + h, \ \ y_0 + h\, f(x_0, y_0)\big)\right]$$

$We\ have \quad x_0 = 0, \ y_0 = 0 \ , \ h = 0.1 \ \ \& \ \ f(x, y) = 1 - y$

$$\therefore \quad y_1(0 + 0.1) = 0 + \left(\frac{0.1}{2}\right) \left[f(0,0) + f\big(0 + 0.1, \ \ 0 + 0.1\, f(0,0)\big)\right]$$

$$\begin{aligned} y_1(0.1) &= 0 + (0.05) \left[f(0,0) + f\big(0 + 0.1, \ \ 0 + 0.1\, f(0,0)\big)\right] \\ &= (0.05) \left[(1 - 0) + f(\ 0.1, \ \ 0.1\,[1 - 0])\right] \\ &= (0.05) \left[(1) + f(\ 0.1, \ \ 0.1\,[1])\right] \\ &= (0.05) \left[1 + f(\ 0.1, \ \ 0.1)\right] \\ &= (0.05) \left[1 + (1 - 0.1)\right] \\ &= (0.05) \left[1 + 0.9\right] \\ &= 0.095 \end{aligned}$$

$$y_1(0.1) = 0.095$$

$$\boldsymbol{y_1(0.1) = 0.095} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Rightarrow \ x_1 = \boldsymbol{0.1} \ \ \& \ \boldsymbol{y_1 = 0.095}$$

To find $\boldsymbol{y(0.2)}$:

Put $n = 1$, equation (1) becomes

$$y_2(x_1 + h) = y_1 + \left(\frac{h}{2}\right) \left[f(x_1, y_1) + f\big(x_1 + h, \ \ y_1 + h\, f(x_1, y_1)\big)\right]$$

$We\ have \quad x_1 = 0.1, \ y_1 = 0.095 \ , \ h = 0.1 \ \ \& \ \ f(x, y) = 1 - y$

$$\therefore \quad y_2(0.1 + 0.1) = 0.095 + \left(\frac{0.1}{2}\right) \left[f(0.1, 0.095) + f\big(0.1 + 0.1, \ \ 0.095 + 0.1\, f(0.1, \ 0.095)\big)\right]$$

$$y_2(0.2) = 0.095 + (0.05) \left[(1 - 0.095) + f(0.1 + 0.1, \ \ 0.095 + 0.1\,[1 - 0.095])\right]$$

$= 0.095 + (0.05)\,[(0.905) + f(\,0.2,\;\; 0.095 + 0.1\,[0.905])]$

$= \; 0.095 + (0.05)\,[(0.905) + f(\,0.2,\;\; 0.1855)]$

$= 0.095 + (0.05)\,[0.905 + (1 - 0.1855)]$

$= 0.095 + (0.05)\,[0.905 + 0.8145]$

$= \; 0.095 + (0.05)\,[1.7195]$

$y_2(0.2) = 0.180975$

$\boldsymbol{y_2(0.2) = 0.180975} \qquad [\boldsymbol{y(x_2) = y_2}] \quad \Longrightarrow \quad x_2 = \mathbf{0.2} \;\&\; \boldsymbol{y_2 = 0.180975}$

To find $y(0.3)$:

Put $n = 2$, equation (1) becomes

$$y_3(x_2 + h) = y_2 + \left(\frac{h}{2}\right)\left[f(x_2, y_2) + f\big(x_2 + h,\;\; y_2 + h\, f(x_2, y_2)\big)\right]$$

$We\; have \quad x_2 = 0.1,\;\; y_2 = 0.180975\;\;,\;\; h = 0.1 \;\;\&\;\; f(x,y) = 1 - y$

$$\therefore \quad y_3(0.2 + 0.1) = 0.095 + \left(\frac{0.1}{2}\right)\left[f(0.2, 0.180975) + f\big(0.2 + 0.1,\;\; 0.180975 + 0.1\, f(0.1,\;\; 0.180975)\big)\right]$$

$y_3(0.3) = 0.180975 + (0.05)\,[(1 - 0.180975) + f(0.3,\;\; 0.180975 + 0.1\,[1 - 0.180975])]$

$= 0.180975 + (0.05)\,[(0.819025) + f(\,0.3,\;\; 0.180975 + 0.1\,[0.819025])]$

$= \; 0.180975 + (0.05)\,[(0.819025) + f(\,0.3,\;\; 0.2628775)]$

$= 0.180975 + (0.05)\,[0.819025 + (1 - 0.2628775)]$

$= 0.180975 + (0.05)\,[0.819025 + 0.7371225]$

$= 0.180975 + (0.05)\,[1.5561475]$

$= 0.180975 + 0.077807375$

$\boldsymbol{y_3(0.3) = 0.258782} \qquad [\boldsymbol{y(x_3) = y_3}] \quad \Longrightarrow \quad \boldsymbol{x_3 = 0.3} \;\&\; \boldsymbol{y_3 = 0.258782}$

Example . 9 : Given $y' = x^2 - y\,,\; y(0) = 1$ Find correct to four decimal places the value of $y(0.1)$ by using Improved Euler's method.

Solution : Given

$$y' = f(x,y) = \frac{dy}{dx} = x^2 - y \quad and \quad y(0) = 1 \quad \Longrightarrow \quad x_0 = 0,\;\; y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$$

To find h :

$Since\;\; y(0) = 0 \quad \Longrightarrow \quad y(x_0) = y_0$

$Also\ we\ need\ to\ find\ \ y(0.1)$

$\Longrightarrow \quad y(x_1) = ?, \ \ [y(0.1) = ?] \qquad \Longrightarrow \ x_1 = 0.1,$

$\therefore \ \ h = x_1 - x_0 = 0.1 - 0.0 = 0.1 \qquad [Difference]$

$\therefore \ \ h = 0.1$

The Improved Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + \left(\frac{h}{2}\right)\left[f(x_n, y_n) + \ f\big(x_n + h, \ \ y_n + h\, f(x_n, y_n)\big)\right] \ \ , \ n = 0, 1, 2, \ldots\ldots} \qquad \ldots\ldots\ldots(1)$$

To find $\boldsymbol{y(0.1)}$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + \left(\frac{h}{2}\right)\left[f(x_0, y_0) + \ f\big(x_0 + h, \ \ y_0 + h\, f(x_0, y_0)\big)\right]$$

$We\ have \ \ \ x_0 = 0, \ \ y_0 = 1 \ \ , \ \ h = 0.1 \ \ \& \ \ f(x, y) = x^2 - y$

$$\therefore \quad y_1(0 + 0.1) = 1 + \left(\frac{0.1}{2}\right)\left[f(0,1) + \ f\big(0 + 0.1, \ \ 1 + 0.1\, f(0,1)\big)\right]$$

$$y_1(0.1) \ = 1 + (0.05)\left[f(0,1) + \ f\big(\ 0.1, \ \ 1 + 0.1\, f(0,1)\big)\right]$$

$$= \ 1 + (0.05)\ [(0^2 - 1) + \ f(\ 0.1, \ \ 1 + \ 0.1\ [0^2 - 1])]$$

$$= \ 1 + (0.05)\ [(-1) + \ f(\ 0.1, \ \ 1 + \ 0.1\ [-1])]$$

$$= 1 + \ (0.05)\ [-1 + \ f(\ 0.1, \ \ 0.9)]$$

$$= 1 + \ (0.05)\ [-1 + \langle (0.1)^2 - (0.9) \rangle\ ]$$

$$= \ \ 1 + (0.05)\ [-1 + \ \langle 0.01 - 0.9 \rangle]$$

$$= \ \ 1 + (0.05)\ [-1 - 0.89]$$

$$= \ \ 1 + (0.05)\ [-1.89]$$

$$= \ \ 1 - 0.0945$$

$$y_1(0.1) \ = 0.9055$$

$$\boldsymbol{y_1(0.1) = 0.9055} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow \ x_1 = \boldsymbol{0.1} \ \ \& \ \boldsymbol{y_1 = 0.9055}$$

Example . 10 :

Using Improved Euler's method find $y \ \ at \ \ x = 0.1 \ \ \& \ at \ \ x = 0.2$ Given $y' = y - \frac{2x}{y} \ \ , \ \ y(0) = 1$

Solution : Given

$$y' = f(x,y) = \frac{dy}{dx} = y - \frac{2x}{y} \quad and \quad y(0) = 1 \quad \Longrightarrow \quad x_0 = 0, \ \ y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$$

To find h :

$$Since \ \ y(0) = 0 \qquad \Longrightarrow \quad y(x_0) = y_0$$

$$Also\ we\ need\ to\ find \quad y(0.1) \ \ \& \ \ y(0.2)$$

$$\Longrightarrow \quad y(x_1) = ?, \ \ [y(0.1) = ?\,] \qquad \Longrightarrow \ \ x_1 = 0.1,$$

$$\therefore \ \ h = x_1 - x_0 = 0.1 - 0.0 = 0.1 \qquad [Difference]$$

$$\therefore \ \ \boldsymbol{h = 0.1}$$

The Improved Euler's formula is

$$\boldsymbol{y_{n+1}(x_n + h) = y_n + \left(\frac{h}{2}\right)\left[f(x_n, y_n) + \ f\big(x_n + h, \ \ y_n + h\, f(x_n, y_n)\big)\right] \ \ , \ \ n = 0, 1, 2, \ldots\ldots} \qquad \ldots\ldots\ldots(\mathbf{1})$$

To find $y(0.1)$:

Put $n = 0$, equation (1) becomes

$$y_1(x_0 + h) = y_0 + \left(\frac{h}{2}\right)\left[f(x_0, y_0) + \ f\big(x_0 + h, \ \ y_0 + h\, f(x_0, y_0)\big)\right]$$

$$We\ have \quad x_0 = 0, \ \ y_0 = 1 \ \ , \ \ h = 0.1 \ \ \& \ \ f(x,y) = x^2 - y$$

$$\therefore \quad y_1(0 + 0.1) = 1 + \left(\frac{0.1}{2}\right)\left[f(0,1) + \ f\big(0 + 0.1, \ \ 1 + 0.1\, f(0,1)\big)\right]$$

$$y_1(0.1) \ = 1 + (0.05)\left[f(0,1) + \ f\big(\ 0.1, \ \ 1 + 0.1\, f(0,1)\big)\right]$$

$$= \ 1 + (0.05)\left[\left(1 - \frac{2(0)}{1}\right) + \ f\left(\ 0.1, \ \ 1 + \ 0.1\left[1 - \frac{2(0)}{1}\right]\right)\right]$$

$$= \ 1 + (0.05)\,[(1) + \ f(\ 0.1, \ \ 1 + \ \ 0.1\,[1])]$$

$$= 1 + \ (0.05)\,[1 + \ f(\ 0.1, \ \ 1.1)]$$

$$= 1 + \ (0.05)\left[1 + \left(1.1 - \frac{2(0.1)}{1.1}\right)\right]$$

$$= \ \ 1 + (0.05)\,[1 + (0.91818)]$$

$$= \ \ 1 + (0.05)\,[1.91818]$$

$$= \ \ 1 + 0.095909$$

$$y_1(0.1) = \ 1.095909$$

$$\boldsymbol{y_1(0.1) = 1.095909} \qquad [\boldsymbol{y(x_1) = y_1}] \quad \Longrightarrow \ \ \boldsymbol{x_1 = 0.1 \ \ \& \ \ y_1 = 1.095909}$$

To find $y(0.2)$:

Put $n=1$, equation (1) becomes

$$y_2(x_1+h) = y_1 + \left(\frac{h}{2}\right)\left[f(x_1,y_1) + f\big(x_1+h,\ \ y_1 + h\, f(x_1,y_1)\big)\right]$$

$We\ have\ \ \ x_1 = 0.1,\ \ y_1 = 1.095909\ \ ,\ \ h = 0.1\ \ \&\ \ f(x,y) = x^2 - y$

$$\therefore\ y_2(0.1+0.1) = 1.095909 + \left(\frac{0.1}{2}\right)\left[f(0.1, 1.095909) + f\big(0.1+0.1,\ \ 1.095909 + 0.1\, f(0.1,1.095909)\big)\right]$$

$$y_2(0.2)\ = 1.095909 + (0.05)\left[f(0.1, 1.095909) + f\big(\ 0.2,\ \ 1.095909 + 0.1\, f(0.1,1.095909)\big)\right]$$

$$= 1.095909 + (0.05)\left[\left(1.095909 - \frac{2(0.1)}{1.095909}\right) + f\left(\ 0.1,\ \ 1+\ 0.1\left[1.095909 - \frac{2(0.1)}{1.095909}\right]\right)\right]$$

$$= 1.095909 + (0.05)\left[(0.91341) + f(\ 0.1,\ \ 1+\ \ 0.1\,[0.91341])\right]$$

$$= 1.095909 +\ (0.05)\left[0.91341 + f(\ 0.1,\ \ 1.091341)\right]$$

$$= 1.095909 +\ (0.05)\left[0.91341 + \left(1.091341 - \frac{2(0.1)}{1.091341}\right)\right]$$

$$= 1.095909 + (0.05)\left[0.91341 + (0.90808)\right]$$

$$= 1.095909 + (0.05)\left[1.821491\right]$$

$$= 1.095909 + 0.091074$$

$$y_2(0.2) = 1.18698$$

$$\boldsymbol{y_2(0.2) = 1.18698} \qquad [y(x_2) = y_2] \Longrightarrow\ \boldsymbol{x_2 = 0.1\ \ \&\ \ y_1 = 1.18698}$$

MILNE'S PREDICTOR CORRCETOR METHOD

Predictor : $\boldsymbol{y_{n+1,P}(x_n + h) = y_{n-3} + \frac{4h}{3}\left[2\, y'_{n-2} - y'_{n-1} + 2\, y'_n\right]}$

Corrector : $\boldsymbol{y_{n+1,C}(x_n + h) = y_{n-1} + \frac{h}{3}\left[\ y'_{n-1}\ + 4y'_n +\ y'_{n+1}\right]}$

Example . 1 :

Given $y' = x^3 + y$, $y(0) = 2$. Also given $y(0.2) = 2.073$, $y(0.4) = 2.452\ \ and\ \ y(0.6) = 3.023$.

Find $y(0.8)$ By Using Milne's Method

Solution : Given $y' = f(x,y) = \frac{dy}{dx} = x^3 + y\ \ \&\ \ y(0) = 2\ \ \Longrightarrow\ x_0 = 0,\ \ y_0 = 2\ \ [Since\ \ y(x_0) = y_0]$

$y(0) = 2$	$y(x_0) = y_0$	$x_0 = 0$	$y_0 = 2$
$y(0.2) = 2.073$	$y(x_1) = y_1$	$x_1 = 0.2$	$y_1 = 2.073$

$y(0.4) = 2.452$	$y(x_2) = y_2$	$x_2 = 0.4$	$y_2 = 2.452$
$y(0.6) = 3.023$	$y(x_3) = y_3$	$x_3 = 0.6$	$y_3 = 3.023$

$\boldsymbol{Here \quad h = 0.2}$ **and** $\boldsymbol{n = 3 \ \ [Highest\ value\ of\ x\ is\ x_3 . \ \ \therefore n = 3]}$

The Milne's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_{n-3} + \frac{4h}{3}[2\,y'_{n-2} - y'_{n-1} + 2\,y'_n] \quad \ldots\ldots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3] \quad \ldots\ldots(2)$$

$Given \quad y' = x^3 + y$

$x_1 = 0.2$	$y_1 = 2.073$	$y'_1 = x_1^3 + y_1$	$y'_1 = (0.2)^3 + (2.073)$	$y'_1 = 2.081$
$x_2 = 0.4$	$y_2 = 2.452$	$y'_2 = x_2^3 + y_2$	$y'_2 = (0.4)^3 + (2.452)$	$y'_2 = 2.516$
$x_3 = 0.6$	$y_3 = 3.023$	$y'_3 = x_3^3 + y_2$	$y'_3 = (0.6)^3 + (3.023)$	$y'_3 = 3.239$

Equation (2) becomes

$$y_{4,P}(0.6 + 0.2) = 2 + \frac{4(0.2)}{3}[2\,(2.081) - (2.516) + 2\,(3.239)]$$

$$y_{4,P}(0.8) \ = 2 + \frac{0.8}{3}[8.124] = 2 + 2.1664$$

$$\boldsymbol{y_{4,P}(0.8) \ = 4.1664 \quad [y(x_4) = y_4, \quad x_4 = 0.8 \ \ \& \ \ y_4 = 4.1664\,]}$$

The Milne's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_{n-1} + \frac{h}{3}[\ y'_{n-1} + 4y'_n + \ y'_{n+1}] \quad \ldots\ldots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_2 + \frac{h}{3}[\ y'_2 + 4y'_3 + \ y'_4] \quad \ldots\ldots(3)$$

$x_4 = 0.8$	$y_4 = 4.1664$	$y'_4 = x_4^3 + y_4$	$y'_4 = (0.8)^3 + (4.1664)$	$y'_4 = 4.6784$

Equation (4) becomes

$$y_{4,C}(0.6 + 0.2) = 2.452 + \frac{(0.2)}{3}[2.516 + 4(3.239) + 4.6784]$$

$$y_{4,C}(0.8) \ = 2.452 + \frac{(0.2)}{3}[20.1504]$$

$$\boldsymbol{y_{4,C}(0.8) \ = 3.79536 \quad [y(x_4) = y_4, \ \ x_4 = 0.8 \ \ \& \ \ y_4 = 3.79536\,]}$$

Result:

$$\boldsymbol{y_{4,P}(0.8) \ = 4.1664 \quad \& \quad y_{4,C}(0.8) \ = 3.79536}$$

Example . 2 :

Determine the value of $y(0.4)$ Using Milne's Method, given $y' = xy + y^2,\ y(0) = 1$.

Use Taylor series to get the values of $y(0.1),\ y(0.2)\ \ \&\ \ y(0.3)$.

Solution :

$$Given \quad y' = xy + y^2 \quad \& \quad y(0) = 1 \ \Longrightarrow\ x_0 = 0, \qquad y_0 = 1 \qquad Since \quad [y(x_0) = y_0]$$

Taylor series formula is

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\ y_0' + \frac{(x - x_0)^2}{2\,!}\ y_0'' + \frac{(x - x_0)^3}{3\,!}\ y_0''' + \ \ldots \quad \ldots..(1)$$

$y' = xy + y^2$	$y_0' = x_0\ y_0 + y_0^2 = 0\ (1) + 1^2$	$y_0' = 1$
$y'' = x\ y' + y + \ 2y\ y'$	$y_0'' = x_0\ y_0' + y_0 + \ 2y_0\ y_0'$ $= 0(+1) + 1 + 2\ (1)(1) = 3$	$y_0'' = 3$
$y''' = x\ y'' + y' + y' + 2yy'' + 2y'y'$ $y''' = x\ y'' + 2y' + 2yy'' + 2(y')^2$	$y_0''' = x_0\ y_0'' + y_0'' + 2y_0y_0'' + 2(y_0')^2$ $= 0(3)\ + 2(1) + 2(1)(3) + 2(1)^2$	$y_0''' = 10$

Therefore equation (1) becomes,

$$y(x) = y_0 + \frac{(x - x_0)}{1\,!}\ y_0' + \frac{(x - x_0)^2}{2\,!}\ y_0'' + \frac{(x - x_0)^3}{3\,!}\ y_0''' \ \ldots \quad \ldots..(1)$$

$$= 1 + \frac{(x - 0)}{1}\ (1) + \frac{(x - 0)^2}{2}\ (3) + \frac{(x - 0)^3}{6}\ (10) + \cdots$$

$$y(x) = 1 + x + \frac{3\,x^2}{2} \ + \frac{5\,x^3}{3}$$

To find $\boldsymbol{y(0.1)\ [y\ \ at\ \ x = 0.1\,]}$

$$\therefore \quad y(0.1) = 1 + 0.1 + \frac{3\,(0.1)^2}{2} + \frac{5\,(0.1)^3}{3} \quad = 1 + 0.1 + 0.015 + 0.0016667$$

$$\therefore \quad \boldsymbol{y(0.1) \ = 1.1167}$$

To find $\boldsymbol{y(0.2)\ [y\ \ at\ \ x = 0.2\,]}$

$\therefore \quad y(0.2) = 1 + 0.2 + \dfrac{3\,(0.2)^2}{2} + \dfrac{5\,(0.2)^3}{3} \quad = 1 + 0.2 + 0.06 + 0.013333$

$\therefore \quad \boldsymbol{y(0.2) \; = 1.2733}$

To find $\boldsymbol{y(0.3)}$ $[\boldsymbol{y \;\; at \;\; x = 0.3}\,]$

$\therefore \quad y(0.3) = 1 + 0.3 + \dfrac{3\,(0.3)^2}{2} + \dfrac{5\,(0.3)^3}{3} \quad = 1 + 0.3 + 0.135 + 0.045$

$\therefore \quad \boldsymbol{y(0.3) \; = 1.4800}$

To find $\boldsymbol{y(0.4)}$ **:** Given

$y' = f(x,y) = \frac{dy}{dx} = xy + y^2 \quad and \quad y(0) = 1 \quad \Longrightarrow \quad x_0 = 0, \;\; y_0 = 1 \qquad [Since \quad y(x_0) = y_0]$

$y(0) = 2$	$y(x_0) = y_0$	$x_0 = 0$	$y_0 = 1$
$y(0.1) = 2.073$	$y(x_1) = y_1$	$x_1 = 0.1$	$y_1 = 1.1167$
$y(0.2) = 2.452$	$y(x_2) = y_2$	$x_2 = 0.2$	$y_2 = 1.2733$
$y(0.3) = 3.023$	$y(x_3) = y_3$	$x_3 = 0.3$	$y_3 = 1.4800$

$\boldsymbol{Here \quad h = 0.1}$ **and** $\boldsymbol{n = 3} \;\; [\boldsymbol{Highest\ value\ of\ x\ is\ x_3\,.\;\; \therefore n = 3}]$

The Milne's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_{n-3} + \frac{4h}{3}[2\,y'_{n-2} - y'_{n-1} + 2\,y'_n] \quad \ldots\ldots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 \; - y'_2 + 2\,y'_3] \quad \ldots\ldots(2)$$

$Given \quad y' = xy + y^2$

$x_1 = 0.1$	$y_1 = 1.1167$	$y'_1 = x_1y_1 + y_1^2$	$y'_1 = (0.1)(1.1167) + (1.1167)^2$	$y'_1 = 1.35869$
$x_2 = 0.2$	$y_2 = 1.2733$	$y'_2 = x_2y_2 + y_2^2$	$y'_2 = (0.2)(1.2733) + (1.2733)^2$	$y'_2 = 1.8759$
$x_3 = 0.3$	$y_3 = 1.4800$	$y'_3 = x_3y_3 + y_3^2$	$y'_3 = (0.3)(1.4800) + (1.4800)^2$	$y'_3 = 2.6344$

Equation (2) becomes

$$y_{4,P}(0.3 + 0.1) = 1 + \frac{4(0.1)}{3}[2\,(1.35869) - (1.8759) + 2\,(2.6344)]$$

$$y_{4,P}(0.4) \;\; = 1 + \frac{0.4}{3}[6.11028] = 1 + 0.814704$$

$$\boldsymbol{y_{4,P}(0.4) \ = 1.8147 \quad [y(x_4) = y_4, \ \ x_4 = 0.4 \ \ \& \ \ y_4 = 1.8147]}$$

The Milne's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_{n-1} + \frac{h}{3}[\ y'_{n-1} + 4y'_n + \ y'_{n+1}] \quad \ldots\ldots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_2 + \frac{h}{3}[\ y'_2 + 4y'_3 + \ y'_4] \quad \ldots\ldots(3)$$

$x_4 = 0.4$	$y_4 = 1.8147$	$y'_4 = \ x_4 y_4 + y_4^2$	$y'_4 = (0.4)(1.8147) + (1.8147)^2$	$y'_4 = 4.01902$

Equation (4) becomes

$$y_{4,C}(0.3 + 0.1) = 1.2733 + \frac{(0.1)}{3}[1.8759 + 4(2.6344) + 4.01902]$$

$$y_{4,C}(0.4) \ = 1.2733 + \frac{(0.1)}{3}[16.43252] = 1.2733 + 0.54775$$

$$\boldsymbol{y_{4,C}(0.4) \ = 1.82105 \quad [y(x_4) = y_4, \ \ x_4 = 0.4 \ \& \ y_4 = 1.82105 \ ]}$$

Result:

$$\boldsymbol{y_{4,P}(0.4) \ = 1.8147 \quad \& \quad y_{4,C}(0.4) \ = 1.82105}$$

Example . 2 :

Using Milne's Method Find $y(4.4)$ Given $5xy' + y^2 - 2 = 0$ *given* $y(4) = 1, \ Y(4.1) = 1.0049,$ $y(4.2) = 1.0097 \ \& \ y(4.3) = 1.0143$

Solution : Given $5xy' + y^2 - 2 = 0 \ \Rightarrow \ 5xy' = 2 - y^2 \quad \Rightarrow \quad y' = \frac{2-y^2}{5x}$

$$y' = f(x, y) = \frac{dy}{dx} = \frac{2 - y^2}{5x}$$

$y(4) = 1$	$y(x_0) = y_0$	$x_0 = 4$	$y_0 = 1$
$y(4.1) = 2.073$	$y(x_1) = y_1$	$x_1 = 4.1$	$y_1 = 1.0049$
$y(4.2) = 2.452$	$y(x_2) = y_2$	$x_2 = 4.2$	$y_2 = 1.0097$
$y(4.3) = 3.023$	$y(x_3) = y_3$	$x_3 = 4.3$	$y_3 = 1.0143$

Here $\boldsymbol{h = 0.1}$ **and** $\boldsymbol{n = 3 \ \ [Highest\ value\ of\ \ x\ \ is\ \ x_3. \ \therefore n = 3]}$

The Milne's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_{n-3} + \frac{4h}{3}[2\, y'_{n-2} - y'_{n-1} + 2\, y'_n] \quad \ldots\ldots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3+h) = y_0 + \frac{4h}{3}[2\,y_1' - y_2' + 2\,y_3'] \quad \dots\dots(2)$$

$$Given \quad y' = \frac{2-y^2}{5x}$$

$x_1 = 4.1$	$y_1 = 1.0049$	$y_1' = \frac{2-y_1^{\,2}}{5x_1}$	$y_1' = \frac{2-(1.0049)^2}{5(4.1)}$	$y_1' = 0.0493$
$x_2 = 4.2$	$y_2 = 1.0097$	$y_2' = \frac{2-y_2^{\,2}}{5x_2}$	$y_2' = \frac{2-(1.0097)^2}{5(4.2)}$	$y_2' = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y_3' = \frac{2-y_3^{\,2}}{5x_3}$	$y_3' = \frac{2-(1.0143)^2}{5(4.3)}$	$y_3' = 0.0452$

Equation (2) becomes

$$y_{4,P}(x_3+h) = y_0 + \frac{4h}{3}[2\,y_1' - y_2' + 2\,y_3']$$

$$y_{4,P}(4.3+0.1) = 1 + \frac{4(0.1)}{3}[2\,(0.0493) - (0.0467) + 2\,(0.0452)]$$

$$y_{4,P}(4.4) \;= 1 + \frac{0.4}{3}[0.1423] = 1 + 0.0189733$$

$$\boldsymbol{y_{4,P}(4.4) \;= 1.01897 \quad [y(x_4) = y_4, \;\; x_4 = 4.4 \;\; \& \;\; y_4 = 1.01897\,]}$$

The Milne's Corrector formula is

$$y_{n+1,C}(x_n+h) = y_{n-1} + \frac{h}{3}[\,y_{n-1}' + 4y_n' + \,y_{n+1}'] \quad \dots\dots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3+h) = y_2 + \frac{h}{3}[\,y_2' + 4y_3' + \,y_4'] \quad \dots\dots(3)$$

$x_4 = 4.4$	$y_4 = 1.01897$	$y_4' = \frac{2-y_4^{\,2}}{5x_4}$	$y_4' = \frac{2-(1.01897\,)^2}{5(4.4)}$	$y_4' = 0.0437$

Equation (4) becomes

$$y_{4,C}(4.3+0.1) = 1.0097 + \frac{(0.1)}{3}[0.0467 + 4(0.0452) + 0.0437]$$

$$y_{4,C}(4.4) \;= 1.0097 + \frac{(0.1)}{3}[0.2712] \;= 1.0097 + 0.00904$$

$$\boldsymbol{y_{4,C}(4.4) \;= 1.01874 \quad [y(x_4) = y_4, \;\; x_4 = 4.4 \;\; \& \;\; y_4 = 1.01874\,]}$$

Result:

$\boldsymbol{y_{4,P}(4.4) = 1.01897}$ & $\boldsymbol{y_{4,C}(4.4) = 1.01874}$

Example . 3 :

Solve $y' = x - y^2,\ 0 \le x \le 1,\ y(0) = 0,\ y(0.2) = 0.02,\ y(0.4) = 0.0795,\ y(0.6) = 0.1762$ by Milne's Method to find $y(0.8)$ & $y(1)$.

Solution : Given $y' = x - y^2$

$y(0) = 1$	$y(x_0) = y_0$	$x_0 = 0$	$y_0 = 0$
$y(0.2) = 0.02$	$y(x_1) = y_1$	$x_1 = 0.2$	$y_1 = 0.02$
$y(0.4) = 0.0795$	$y(x_2) = y_2$	$x_2 = 0.4$	$y_2 = 0.0795$
$y(0.6) = 0.1762$	$y(x_3) = y_3$	$x_3 = 0.6$	$y_3 = 0.1762$

Here $\boldsymbol{h = 0.2}$ **and** $\boldsymbol{n = 3}$ $[\boldsymbol{Highest\ value\ of\ x\ is\ x_3.\ \therefore n = 3}]$

The Milne's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_{n-3} + \frac{4h}{3}[2\,y'_{n-2} - y'_{n-1} + 2\,y'_n] \quad \ldots\ldots(1)$$

To Find y(0.8) :

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3] \quad \ldots\ldots(2)$$

Given $y' = x - y^2$

$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = x_1 - y_1^2$	$y'_1 = (0.2) - (0.02)^2$	$y'_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = x_2 - y_2^2$	$y'_2 = (0.4) - (0.0795)^2$	$y'_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = x_3 - y_3^2$	$y'_3 = (0.6) - (0.1762)^2$	$y'_3 = 0.5690$

Equation (2) becomes

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3]$$

$$y_{4,P}(0.6 + 0.2) = 0 + \frac{4(0.2)}{3}[2\,(0.1996) - (0.3937) + 2\,(0.5690)]$$

$$y_{4,P}(0.8) = \frac{0.8}{3}[1.1435]$$

$$\boldsymbol{y_{4,P}(0.8) = 0.3049} \quad [y(x_4) = y_4,\ \ x_4 = 0.8\ \ \&\ \ y_4 = 0.3049]$$

The Milne's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_{n-1} + \frac{h}{3}[\ y'_{n-1} + 4y'_n + \ y'_{n+1}] \quad \dots\dots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_2 + \frac{h}{3}[\ y'_2 + 4y'_3 + \ y'_4] \quad \dots\dots(3)$$

$x_4 = 0.8$	$y_4 = 0.3049$	$y'_4 = x_4 - y_4^2$	$y'_4 = \ 0.8 - (0.3049)^2$	$y'_4 = 0.707$

Equation (4) becomes

$$y_{4,C}(0.6 + 0.2) = 0.0795 + \frac{(0.2)}{3}[0.3937 + 4(0.5690) + 0.707]$$

$$y_{4,C}(0.8) \ = 0.07957 + \frac{(0.2)}{3}[3.376]$$

$$\boldsymbol{y_{4,C}(0.8) \ = 0.3046 \quad [y(x_4) = y_4, \ \ x_4 = 0.8 \ \ \& \ \ y_4 = 0.3046\,]}$$

To Find y(1.0) :

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3] \quad \dots\dots(2)$$

$Given \quad y' = x - y^2$

$x_1 = 0.2$	$y_1 = 0.02$	$y'_1 = x_1 - y_1^2$	$y'_1 = (0.2) - (0.02)^2$	$y'_1 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0795$	$y'_2 = \ x_2 - y_2^2$	$y'_2 = (0.4) - (0.0795)^2$	$y'_2 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y'_3 = \ x_3 - y_3^2$	$y'_3 = (0.6) - (0.1762)^2$	$y'_3 = 0.5690$

Equation (2) becomes

$$y_{4,P}(x_3 + h) = y_0 + \frac{4h}{3}[2\,y'_1 - y'_2 + 2\,y'_3]$$

$$y_{4,P}(0.6 + 0.2) = 0 + \frac{4(0.2)}{3}[2\,(0.1996) - (0.3937) + 2\,(0.5690)]$$

$$y_{4,P}(0.8) \ = \frac{0.8}{3}[1.1435]$$

$$\boldsymbol{y_{4,P}(0.8) \ = 0.3049 \quad [y(x_4) = y_4, \ \ x_4 = 0.8 \ \ \& \ \ y_4 = 0.3049\,]}$$

The Milne's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_{n-1} + \frac{h}{3}[\ y'_{n-1} + 4y'_n + \ y'_{n+1}] \quad \dots\dots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_2 + \frac{h}{3}[\ y_2' \ + 4y_3' + \ y_4'] \quad \ldots\ldots(3)$$

$x_4 = 0.8$	$y_4 = 0.3049$	$y_4' = x_4 - y_4^2$	$y_4' = \ \ 0.8 - (0.3049)^2$	$y_4' = 0.707$

Equation (4) becomes

$$y_{4,C}(0.6 + 0.2) = 0.0795 + \frac{(0.2)}{3}[0.3937 + 4(0.5690) + 0.707]$$

$$y_{4,C}(0.8) \ \ = 0.07957 + \frac{(0.2)}{3}[3.376]$$

$$\boldsymbol{y_{4,C}(0.8) \ \ = 0.3046 \quad [y(x_4) = y_4, \ \ x_4 = 0.8 \ \ \& \ \ y_4 = 0.3046\,]}$$

Result:

$$\boldsymbol{y_{4,P}(0.8) \ \ = 0.3049 \quad \& \quad y_{4,C}(0.8) \ \ = 0.3046}$$

ADAMM'S BASHFORTH PREDICTOR & CORRECTOR METHOD

$\boldsymbol{Predictor}$: $$\boldsymbol{y_{n+1,P}(x_n + h) = y_n + \frac{h}{24}[55y_n' - 59y_{n-1}' + 37\ y_{n-2}' - 9\ y_{n-3}']}$$

$\boldsymbol{Corrector}$: $$\boldsymbol{y_{n+1,C}(x_n + h) = y_n + \frac{h}{24}[9\ y_{n+1}' \ + 19y_n' - 5y_{n-1}' + \ y_{n-2}']}$$

Example . 1 :

Given $\frac{dy}{dx} = x^2(1 + y)$, $y(1) = 1$. Also given $y(1.1) = 1.233$, $y(1.2) = 1.548$ and $y(1.3) = 1.979$.

Find $y(1.4)$ By Using Adam's Method.

Solution : Given

$$y' = f(x, y) = \frac{dy}{dx} = x^2(1 + y)$$

$y(1) = 1$	$y(x_0) = y_0$	$x_0 = 1$	$y_0 = 1$
$y(1.1) = 1.233$	$y(x_1) = y_1$	$x_1 = 1.1$	$y_1 = 1.233$
$y(1.2) = 1.548$	$y(x_2) = y_2$	$x_2 = 1.2$	$y_2 = 1.548$
$y(1.3) = 1.979$	$y(x_3) = y_3$	$x_3 = 1.3$	$y_3 = 1.979$

$\boldsymbol{Here \ \ h = 0.1}$ **and** $\boldsymbol{n = 3 \ \ [Highest\ value\ of\ x\ is\ x_3.\ \ \therefore n = 3]}$

The Adam's Predictor formula is

$$y_{n+1,P}(x_n+h) = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37\, y'_{n-2} - 9\, y'_{n-3}] \quad \ldots\ldots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3+h) = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37\, y'_1 - 9\, y'_0] \quad \ldots\ldots(2)$$

$Given \quad y' = x^2(1+y)$

$x_0 = 1$	$y_0 = 1$	$y'_0 = x_0^2(1+y_0)$	$y'_0 = (1)^2(1+1)$	$y'_1 = 2$
$x_1 = 1.1$	$y_1 = 1.233$	$y'_1 = x_1^2(1+y_1)$	$y'_1 = (1.1)^2(1+1.233)$	$y'_1 = 2.70193$
$x_2 = 1.2$	$y_2 = 1.548$	$y'_2 = x_2^2(1+y_2)$	$y'_2 = (1.2)^2(1+1.548)$	$y'_2 = 3.66912$
$x_3 = 1.3$	$y_3 = 1.979$	$y'_3 = x_3^2(1+y_3)$	$y'_3 = (1.3)^2(1+1.979)$	$y'_3 = 2.0345$

Equation (2) becomes

$$y_{4,P}(1.3+0.1) = 1.979 + \frac{0.1}{24}[55(2.0345) - 59(3.66912) + 37\,(2.70193) - 9\,(2)]$$

$$y_{4,P}(1.4) \;= 1.979 + \frac{0.1}{24}[142.33683] = 1.979 + 0.593070$$

$$\boldsymbol{y_{4,P}(1.4) \;= 2.5721 \quad [y(x_4) = y_4,\;\; x_4 = 1.4 \;\;\&\;\; y_4 = 2.5721\,]}$$

The Adams's Corrector formula is

$$y_{n+1,C}(x_n+h) = y_n + \frac{h}{24}[\,9\, y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}] \quad \ldots\ldots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3+h) = y_3 + \frac{h}{24}[\,9\, \boldsymbol{y'_4} + 19y'_3 - 5y'_2 + y'_1] \quad \ldots\ldots(4)$$

$x_4 = 1.4$	$y_4 = 2.5721$	$y'_4 = x_4^2(1+y_4)$	$y'_4 = (1.4)^2(1+1.2751)$	$y'_4 = 7.7030716$

Equation (4) becomes

$$y_{4,C}(1.3+0.1) = 1.979 + \frac{0.1}{24}[\,9\,(7.7030716) + 19\,(5.0345) - 5\,(3.60912) + (2.70193)]$$

$$y_{4,C}(0.8) \;= 1.979 + \frac{0.1}{24}[143.58827] = 1.979 + 0.592844$$

$$\boldsymbol{y_{4,C}(0.8) \;= 2.57728 \quad [y(x_4) = y_4,\;\; x_4 = 1.4 \;\;\&\;\; y_4 = 2.57728\,]}$$

Result:

$$\boldsymbol{y_{4,P}(1.4) \;= 2.5721 \quad \& \quad y_{4,C}(1.4) \;= 2.5778}$$

Example . 2 :

Using Adam's Method Find $y(4.4)$ Given $5xy' + y^2 - 2 = 0$ *given* $y(4) = 1,\ Y(4.1) = 1.0049,$ $y(4.2) = 1.0097$ & $y(4.3) = 1.0143$

Solution : Given $5xy' + y^2 - 2 = 0 \Rightarrow 5xy' = 2 - y^2 \Rightarrow y' = \frac{2-y^2}{5x}$

$$y' = f(x,y) = \frac{dy}{dx} = \frac{2-y^2}{5x}$$

$y(4) = 1$	$y(x_0) = y_0$	$x_0 = 4$	$y_0 = 1$
$y(4.1) = 2.073$	$y(x_1) = y_1$	$x_1 = 4.1$	$y_1 = 1.0049$
$y(4.2) = 2.452$	$y(x_2) = y_2$	$x_2 = 4.2$	$y_2 = 1.0097$
$y(4.3) = 3.023$	$y(x_3) = y_3$	$x_3 = 4.3$	$y_3 = 1.0143$

Here $\boldsymbol{h = 0.1}$ **and** $\boldsymbol{n = 3}$ $[\boldsymbol{Highest\ value\ of\ x\ is\ x_3.\ \therefore n = 3}]$

The Adam's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37\,y'_{n-2} - 9\,y'_{n-3}] \quad \dots\dots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3 + h) = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37\,y'_1 - 9\,y'_0] \quad \dots\dots(2)$$

Given $y' = \frac{2-y^2}{5x}$

$x_0 = 4$	$y_0 = 1$	$y'_0 = \frac{2-y_1^2}{5x_1}$	$y'_0 = \frac{2-(1)^2}{5(4)}$	$y'_1 = 0.05$
$x_1 = 4.1$	$y_1 = 1.0049$	$y'_1 = \frac{2-y_1^2}{5x_1}$	$y'_1 = \frac{2-(1.0049)^2}{5(4.1)}$	$y'_1 = 0.0483$
$x_2 = 4.2$	$y_2 = 1.0097$	$y'_2 = \frac{2-y_2^2}{5x_2}$	$y'_2 = \frac{2-(1.0097)^2}{5(4.2)}$	$y'_2 = 0.0467$
$x_3 = 4.3$	$y_3 = 1.0143$	$y'_3 = \frac{2-y_3^2}{5x_3}$	$y'_3 = \frac{2-(1.0143)^2}{5(4.3)}$	$y'_3 = 0.0452$

Equation (2) becomes

$$y_{4,P}(x_3 + h) = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37\,y'_1 - 9\,y'_0]$$

$$y_{4,P}(4.3 + 0.1) = 1.0143 + \frac{0.1}{24}[55\,(0.0452) - 59\,(0.0467) + 37\,(0.0483) - 9\,(0.05)]$$

$$y_{4,P}(4.4) = 1.0143 + \frac{0.1}{24}[1.0678] = 1.0186$$

$$y_{4,P}(4.4) \;= 1.0186 \quad [y(x_4) = y_4, \quad x_4 = 4.4 \quad \& \quad y_4 = 1.0186]$$

The Adam's Corrector formula is

$$y_{n+1,C}(x_n + h) = y_n + \frac{h}{24}[\,9\, y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}] \quad \ldots\ldots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3 + h) = y_3 + \frac{h}{24}[\,9\, \boldsymbol{y'_4} + 19y'_3 - 5y'_2 + y'_1] \quad \ldots\ldots(3)$$

$x_4 = 4.4$	$y_4 = 1.0186$	$y'_4 = \frac{2 - y_4^{\,2}}{5x_4}$	$y'_4 = \frac{2 - (1.0186\,)^2}{5(4.4)}$	$y'_4 = 0.0437$

Equation (4) becomes

$$y_{4,C}(4.3 + 0.1) = 1.0143 + \frac{0.1}{24}[\,9\,(0.0437) + 19\,(0.0452) - 5\,(0.0467) + \,(0.0483)]$$

$$y_{4,C}(4.4) \;= 1.979 + \frac{0.1}{24}[1.0669]$$

$$y_{4,C}(4.4) \;= 1.0187 \quad [y(x_4) = y_4, \quad x_4 = 4.4 \quad \& \quad y_4 = 1.0187]$$

Result:

$$y_{4,P}(4.4) \;= 1.0186 \quad \& \quad y_{4,C}(4.4) \;= 1.0187$$

Example . 3 :

Given $\frac{dy}{dx} = \frac{1}{2}(1 + x)\, y^2$, $y(0) = 1$. Also given $y(0.1) = 1.0546$, $y(0.2) = 1.1227$ and $y(0.3) = 1.2074$.

Find $y(0.4)$ By Using Adam's Method.

Solution : Given

$$y' = f(x,y) = \frac{dy}{dx} = \frac{1}{2}(1 + x)\, y^2$$

$y(0) = 1$	$y(x_0) = y_0$	$x_0 = 0$	$y_0 = 1$
$y(0.1) = 1.0456$	$y(x_1) = y_1$	$x_1 = 0.1$	$y_1 = 1.0456$
$y(0.2) = 1.1277$	$y(x_2) = y_2$	$x_2 = 0.2$	$y_2 = 1.1277$
$y(0.3) = 1.2074$	$y(x_3) = y_3$	$x_3 = 0.3$	$y_3 = 1.2074$

Here $\boldsymbol{h = 0.1}$ **and** $\boldsymbol{n = 3}$ $[\boldsymbol{Highest\ value\ of\ x\ is\ x_3.\ \therefore n = 3}]$

The Adam's Predictor formula is

$$y_{n+1,P}(x_n + h) = y_n + \frac{h}{24}\,[55y'_n - 59y'_{n-1} + 37\, y'_{n-2} - 9\, y'_{n-3}] \quad \ldots\ldots(1)$$

Put n=3 in equation (1), we have

$$y_{4,P}(x_3+h) = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37\, y_1' - 9\, y_0'] \qquad \dots\dots(2)$$

$$Given \quad y' = \frac{1}{2}(1+x)\, y^2$$

$x_0 = 0$	$y_0 = 1$	$y_0' = \frac{1}{2}(1+x_0)\, y_0^2$	$y_0' = \frac{1}{2}[1+0]\,(1)^2$	$y_0' = 0.5$
$x_1 = 0.1$	$y_1 = 1.0456$	$y_1' = \frac{1}{2}(1+x_1)\, y_1^2$	$y_1' = \frac{1}{2}[1+0.1]\,(1.0456)^2$	$y_1' = 0.61171$
$x_2 = 0.2$	$y_2 = 1.1227$	$y_2' = \frac{1}{2}(1+x_2)\, y_2^2$	$y_2' = \frac{1}{2}[1+0.2]\,(1.1277)^2$	$y_2' = 0.7563$
$x_3 = 0.3$	$y_3 = 1.2074$	$y_3' = \frac{1}{2}(1+x_3)\, y_3^2$	$y_3' = \frac{1}{2}[1+0.3]\,(1.2074)^2$	$y_3' = 0.9475$

Equation (2) becomes

$$y_{4,P}(0.3+0.1) = 1.2063 + \frac{0.1}{24}[55(0.9475) - 59(0.7563) + 37\,(0.61171) - 9\,(0.5)]$$

$$y_{4,P}(0.4) \;\; = 1.979 + \frac{0.1}{24}[25.5361] = 1.2063 + 0.1064$$

$$\boldsymbol{y_{4,P}(0.4) \;\; = 1.3127} \quad [\boldsymbol{y(x_4) = y_4}, \;\; \boldsymbol{x_4 = 0.4} \;\; \& \;\; \boldsymbol{y_4 = 1.3127}]$$

The Adams's Corrector formula is

$$y_{n+1,C}(x_n+h) = y_n + \frac{h}{24}[\,9\, y_{n+1}' + 19y_n' - 5y_{n-1}' + \, y_{n-2}'] \quad \dots\dots(3)$$

Put n=3 in equation (3), we have

$$y_{4,C}(x_3+h) = y_3 + \frac{h}{24}[\,9\, \boldsymbol{y_4'} + 19y_3' - 5y_2' + \, y_1'] \quad \dots\dots(4)$$

$x_4 = 0.4$	$y_4 = 1.3127$	$y_4{}' = \frac{1}{2}(1+x_4)\, y_4^2$	$y_4{}' = \frac{1}{2}[1+0.4]\,(1.3127)^2$	$y_4{}' = 1.2062$

Equation (4) becomes

$$y_{4,C}(0.3+0.1) = 1.2063 + \frac{0.1}{24}[\,9\,(1.2062) + 19\,(0.9475) - 5\,(0.7563) + \,(0.61171)]$$

$$y_{4,C}(0.4) \;\; = 1.2063 + \frac{0.1}{24}[25.6885]$$

$$\boldsymbol{y_{4,C}(0.4) \;\; = 1.3133} \quad [\boldsymbol{y(x_4) = y_4}, \;\; \boldsymbol{x_4 = 0.4} \;\; \& \;\; \boldsymbol{y_4 = 1.3133}]$$

Result:

$$\boldsymbol{y_{4,P}(0.4) \;\; = 1.3127} \quad \& \quad \boldsymbol{y_{4,C}(0.4) \;\; = 1.3133}$$