PARABOLIC EQUATIONS

BENDER SCHMIDIT METHOD :

The one dimensional Heat equation $\frac{\partial^2 u}{\partial x^2} = a\,\frac{\partial u}{\partial t}$

This is an example of parabolic equations

AIM :

To solve $u_{xx} = a\,u_t$ $\qquad \ldots\ldots(1)$

with boundary conditions $u(0,t) = T_0$ $\qquad \ldots\ldots(2)$

$u(l,t) = T_l$ $\qquad \ldots\ldots(3)$

$u(x,0) = f(x)\ ,\quad 0 < x < l$ $\quad \ldots\ldots(4)$

$u_{i,j+1} = \lambda\,u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda\,u_{i-1,j}$ $\ldots\ldots(5)$

Equation (5) is called **EXPLICIT FORMULA**.

This formula is valid only if $0 < \lambda \le \frac{1}{2}$.

If we take $\lambda = \frac{1}{2}$ then Equation (5) becomes

$$u_{i,j+1} = \frac{1}{2}[\lambda\,u_{i-1,j} + u_{i+1,j}] \quad \ldots\ldots(6)$$

In this case $\lambda = \frac{1}{2}$ *and* $k = \frac{a\,h^2}{2}$

Equation (6) is called **BENDER SCHMIDIT RECCURENCE EQUATION**.

Example . 1: Solve $\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial t} = 0$ given $u(0,t) = 0,\ u(4,t) = 0,\ u(x,0) = x(4-x)$. Assume $h = 1$. Find the values of u *upto* $t = 5$.

Solution :

Given $\frac{\partial^2 u}{\partial x^2} - 2\,\frac{\partial u}{\partial t} = 0 \quad \Rightarrow \quad a = 2$. Also $h = 1$ $[Given]$

Since $k = \frac{a\,h^2}{2} \quad \Rightarrow \quad k = 2\frac{(1)^2}{2} = 1 \quad \Rightarrow k = 1$

Given $\boldsymbol{u(0,t) = 0}$, $\Rightarrow u(0,0) = (0,1) = u(0,2) = u(0,3) = u(0,4) = u(0,5) = 0$

Also $\boldsymbol{u(4,t) = 0}$, $\Rightarrow u(4,0) = (4,1) = u(4,2) = u(4,3) = u(4,0) = u(5,0) = 0$

Since $\boldsymbol{u(x,0) = x(4-x)} \quad \Rightarrow \quad u(0,0) = 0(4-0) = 0$

$$\boldsymbol{u}(\mathbf{1},\mathbf{0}) = 1(4-1) = 3 \;, \quad \boldsymbol{u}(\mathbf{2},\mathbf{0}) = 2(4-2) = 4$$

$$\boldsymbol{u}(\mathbf{3},\mathbf{0}) = 3(4-3) = 3 \;, \quad \boldsymbol{u}(\mathbf{4},\mathbf{0}) = 4(4-4) = 0$$

The values of $u_{i,j}$ are tabulated below

	$\boldsymbol{x-direction} \longrightarrow (\boldsymbol{h-diff})$						
$\boldsymbol{t-direction} \downarrow (\boldsymbol{k-diff})$	$\boldsymbol{j \setminus i}$	**0**	**1**	**2**	**3**	**4**	
	0	0	3	4	3	0	$u(x,0) = x(4-x)$
	1	0	2	3	2	0	
	2	0	1.5	2	1.5	0	
	3	0	1	1.5	1	0	
	4	0	0.75	1	0.75	0	
	5	0	0.5	0.75	0.5	0	

Example. 2: Solve $u_t = u_{xx}$ subject to $u(0,t) = 0, u(1,t) = 0 \;\; and \;\; u(x,0) = \sin\pi x \,, 0 < x < 1$

Solution :

Given $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \Longrightarrow \quad a = 1.$

Since h & k are not given.

Since $k = \frac{a\,h^2}{2} \quad \Longrightarrow \quad k = 1\frac{h^2}{2} = \frac{h^2}{2} \quad Since\ the\ range\ is\ (0{,}1).$

Let us choose $h = 0.2.$

$$\therefore \; k = \frac{(0.2)^2}{2} = \frac{0.04}{2} = 0.02$$

Given $\boldsymbol{u}(\mathbf{0},\boldsymbol{t}) = \mathbf{0}, \;\Longrightarrow u(0{,}0) = (0{,}0.02) = u(0{,}0.04) = u(0{,}0.06) = u(0{,}0.08) = u(0{,}0.1) = 0$

$Also \quad \boldsymbol{u}(\mathbf{1},\boldsymbol{t}) = \mathbf{0}, \;\Longrightarrow u(1{,}0) = (1{,}0.02) = u(1{,}0.04) = u(1{,}0.06) = u(1{,}0.08) = u(1{,}0.1) = 0$

Since $\boldsymbol{u}(\boldsymbol{x},\mathbf{0}) = \mathbf{sin}\,\boldsymbol{\pi x} \quad \Longrightarrow \quad u(0{,}0) = \sin\pi(0) = \sin 0 = 0$

$$\boldsymbol{u}(\mathbf{0.2},\mathbf{0}) = \sin\pi(0.2) = 0.5878 \;, \quad \boldsymbol{u}(\mathbf{0.4},\mathbf{0}) = \sin\pi(0.4) = 0.9511$$

$$\boldsymbol{u}(\mathbf{0.6},\mathbf{0}) = \sin\pi(0.6) = 0.9511 \;, \quad \boldsymbol{u}(\mathbf{0.8},\mathbf{0}) = \sin\pi(0.8) = 0.5878$$

$$\boldsymbol{u}(\mathbf{1.0},\mathbf{0}) = \sin\pi(1) = 0$$

The values of $u_{i,j}$ are tabulated below

	$x - direction \quad \longrightarrow \quad (h - diff)$							
t – *direction* ↓ $(k - diff)$	$j \backslash i$	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**	
	0	0	0.5878	0.9511	0.9511	0.5878	0	$u(x,0) = \sin \pi x$
	0.02	0	0.4756	0.7695	0.7695	0.4756	0	
	0.04	0	0.3848	0.6225	0.6225	0.3848	0	
	0.06	0	0.3113	0.5036	0.5036	0.3113	0	
	0.08	0	0.2518	0.4074	0.4074	0.2518	0	
	0.10	0	0.2037	0.3296	0.3296	0.2037	0	
		$u(0,t) = 0$					$u(0,t) = 0$	

Example . 3: Solve $32\, u_t = u_{xx}$ taking $h = 0.25$ for $t > 0$, $0 < x < 1$ and $u(x,0) = 0$, $u(0,t) = 0,\ u(1,t) = t$.

Solution :

Given $u_{xx} = 32\, u_t \implies a = 32.$ Also $h = 0.25$

Since $k = \frac{a\, h^2}{2} \implies k = 32\, \frac{(0.25)^2}{2} = 1$.

Given $\boldsymbol{u(0,t) = 0}$, $\implies u(0,0) = (0,1) = u(0,2) = u3 = u(0,4) = u(0,5) = 0$

Also $\boldsymbol{u(1,t) = t}$, $\implies u(1,0) = 0,\ u\,(1,1) = 1,\ u(1,2) = 2,\ u(1,3) = 3,\ u(1,4) = 4,\ u(1,5) = 5$

Since $\boldsymbol{u(x,0) = 0} \implies \boldsymbol{u(0,0)} = 0,\quad \boldsymbol{u(0.25,0)} = 0,\quad \boldsymbol{u(0.50,0)} = 0,$

$\boldsymbol{u(0.75,0)} = 0,\quad \boldsymbol{u(1.0,0)} = 0$

The values of $u_{i,j}$ are tabulated below

	$x - direction \quad \longrightarrow \quad (h - diff)$						
t – *direction* ↓ $(k - diff)$	$j \backslash i$	**0**	**0.25**	**0.50**	**0.75**	**1.0**	
	0	0	0	0	0	0	$u(x,0) = 0$

	1	0	0	0	0	1	
	2	0	0	0	0.5	2	
	3	0	0	0.25	1	3	
	4	0	0.125	0.5	1.625	4	
	5	0	0.25	0.875	2.25	5	
		$u(0,t) = 0$				$u(1,t) = t$	

Example . 4: Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $0 \leq x \leq 1$, $t \geq 0$ with $u(x,0) = x(1-x),\ 0 < x < 1\ \ and$

$u(0,t) = u(1,t) = 0\ \ \forall\ t > 0$ using Explicit method with $\Delta x = 0.2$ for three time steps.

Solution :

Given $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \implies a = 1.$ Also $\Delta x = h = 0.2$

Since $k = \frac{a\,h^2}{2} \implies k = \frac{(0.2)^2}{2} = 0.02.$

$$u_{i,j+1} = \frac{1}{2}[\lambda\, u_{i-1,j} + u_{i+1,j}]$$

Given $\boldsymbol{u(0,t) = 0}, \implies u(0,0) = (0,1) = u(0,2) = u3 = u(0,4) = u(0,5) = 0$

$Also$ $\boldsymbol{u(1,t) = 0}, \implies u(1,0) = 0\ ,\ u\ (1,1) = 0,\ u(1,2) = 0\ ,\ u(1,3) = 0\ ,\ u(1,4) = 0,\ u(1,5) = 0$

Since $\boldsymbol{u(x,0) = x(1-x)} \implies \boldsymbol{u(0,0)} = 0\ ,\quad \boldsymbol{u(0.2,0)} = 0.2(1-0.2) = 0.16\ ,$

$\boldsymbol{u(0.4,0)} = 0.4(1-0.4) = 0.24\ ,\quad \boldsymbol{u(0.6,0)} = 0.24\ ,\quad \boldsymbol{u(0.8,0)} = 0.16$

The values of $u_{i,j}$ are tabulated below

	$\boldsymbol{x-direction \longrightarrow (h-diff)}$							
$\boldsymbol{t-direction}\downarrow (\boldsymbol{k-diff})$	$\boldsymbol{j \setminus i}$	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**	
	0	0	0.16	0.24	0.24	0.16	0	$u(x,0) = 0$
	0.2	0	0.12	0.2	0.2	0.12	0	
	0.4	0	0.1	0.16	0.16	0.1	0	

	0.6	0	0.08	0.13	0.13	0.08	0	
	0.8	0	0.065	0.105	0.105	0.065	0	
	1.0	0	0.0525	0.085	0.085	0.0525	0	
		$u(0,t) = 0$					$u(1,t) = 0$	

CRANK-NICOLSON'S METHOD

DIFFERENCE EQUATION CORRESPONDING TO THE PARABOLIC EQUATION

$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ (*IMPILICT METHOD*)

The Crank Nicolson's difference equation in the general form is given by

$$-\lambda u_{i-1,j+1} + (2\lambda + 2)u_{i,j+1} - \lambda u_{i+1,j+1} = \lambda u_{i-1,j} + (2 - 2\lambda)u_{i,j} + \lambda\, u_{i+1,j}$$

If $\lambda = 1,$ the Crank Nicolson's difference equation is takes the form

$$-u_{i-1,j+1} + 4\,u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

$$u_{i,j+1} - u_{i+1,j+1} = u_{i-1,j} + u_{i+1,j}$$

Also $k = ah^2$

Example:

Solve by Crank – Nicholson method the equation $u_{xx} = u_t$ subject to $u(x,0) = 0,\ u(0,t) = 0$ and $u(1,t) = t$, for two time steps.

Solution:

Here x ranges from $0\ to\ 1$, take $h = 1/4$. $a = 1$

$$\therefore \quad k = ah^2, \quad k = 1\left(\frac{1}{4}\right)^2 = \frac{1}{16}.$$

We use the formula

$$u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}\right] - - - (1)$$

	0	0.25	0.5	0.75	1
0	0	0	0	0	0
$\frac{1}{16}$	0	u_1	u_2	u_3	$\frac{1}{16}$
$\frac{2}{16}$	0	u_4	u_5	u_6	$\frac{2}{16}$

$u_1 = \frac{1}{4}[0+0+0+u_2]$	$4u_1 = u_2$	$4u_1 - u_2 = 0$
$u_2 = \frac{1}{4}[0+0+u_1+u_3]$	$4u_2 = [u_1 + u_3]$	$-u_1 + 4u_2 - u_3 = 0$
$u_3 = \frac{1}{4}\left[0+0+u_2+\frac{1}{16}\right]$	$4u_3 = u_2 + \frac{1}{16}$	$0u_1 - u_2 + 4u_3 = \frac{1}{16}$

Solving we get $\boldsymbol{u_1 = 0.0011}$, $\boldsymbol{u_2 = 0.0045}$, $\boldsymbol{u_3 = 0.0168}$

$u_4 = \frac{1}{4}[0+0+u_2+u_5]$	$4u_4 = u_2 + u_5$	$4u_4 - u_5 = 0.0045$
$u_5 = \frac{1}{4}[u_1+u_4+u_3+u_6]$	$4u_5 = 0.0011 + u_4 + 0.0168 + u_6$	$-u_4 + 4u_5 - u_6 = 0.0179$
$u_6 = \frac{1}{4}\left[u_2+u_5+\frac{1}{16}+\frac{2}{16}\right]$	$4u_6 = 0.0045 + u_5 + \frac{3}{16}$	$-u_5 + 4u_6 = 0.192$

Solving we get $\boldsymbol{u_4 = 0.005899}$, $\boldsymbol{u_5 = 0.01913}$, $\boldsymbol{u_6 = 0.052777}$

Example:

Using Crank – Nicholson method, solve the equation $u_{xx} = 16u_t$ subject to $u(x,0) = 0$, $u(0,t) = 0$ and $u(1,t) = 100t$. Compute t for one time step taking $h = 1/4$.

Solution:

Given $h = 1/4$. $a^2 = 16$, $a = 4$

$$\therefore \quad k = ah^2, \quad k = 16\left(\frac{1}{4}\right)^2 = 1.$$

$$We\,use\,the\,formula \quad u_{i,j+1} = \frac{1}{4}\left[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j} + u_{i+1,j}\right] - - - (1)$$

	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	u_1	u_2	u_3	100

$u_1 = \frac{1}{4}[0+0+0+u_2]$	$4u_1 = u_2$	$4u_1 - u_2 = 0$
$u_2 = \frac{1}{4}[0+0+u_1+u_3]$	$4u_2 = [u_1 + u_3]$	$-u_1 + 4u_2 - u_3 = 0$
$u_3 = \frac{1}{4}[0+0+u_2+100]$	$4u_3 = u_2 + 100$	$0u_1 - u_2 + 4u_3 = 100$

Solving we get $\boldsymbol{u_1 = 1.7857,\quad u_2 = 7.1429,\quad u_3 = 26.7857}$

ONE DIMENSIONAL WAVE EQUATION

The wave equation $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ is the simplest form of the hyperbolic equations.

The explicit formula for solving wave equation is given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad where\ k = h/a$$

Example: 1 Solve $y_{tt} = y_{xx}$ upto $t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0,\ y(1,t) = 0$, $y_t(x,0) = 0\ \ and\ \ y(x,0) = 10 + x(10-x)$.

Solution:

Given $y_{tt} = y_{xx} \implies \frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2} \Rightarrow a^2 = 1 \Rightarrow a = 1.$

$$h = 0.1 \qquad \Rightarrow k = \frac{h}{a} = \frac{0.1}{1} = 0.1$$

The explicit formula for solving wave equation is given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$y(x,0) = 10 + x(10-x)$

$$y(0.1,0) = 0.1 + 0.1(10-0.1) = 10.09$$

$$y(0.2,0) = 0.2 + 0.2(10-0.2) = 10.16$$

$$y(0.3,0) = 0.3 + 0.3(10-0.3) = 10.21$$

$$y(0.4,0) = 0.4 + 0.4(10-0.4) = 10.24$$

$$y(0.5,0) = 0.5 + 0.5(10-0.5) = 10.25$$

$$y(0.6,0) = 0.6 + 0.6(10-0.6) = 10.24$$

$$y(0.7,0) = 0.7 + 071(10-0.7) = 10.21$$

$y(0.8,0) = 0.8 + 0.8(10 - 0.8) = 10.16$

$y(0.9,0) = 0.9 + 0.9(10 - 0.9) = 10.09$

x/t	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	0	10.09	10.16	10.21	10.24	10.25	10.24	10.21	10.16	10.09	0
0.1	0	5.08	10.15	10.2	10.23	10.24	10.23	10.2	10.15	5.08	0
0.2	0	0.06	5.12	10.17	10.2	10.21	10.2	10.17	7.64	5.08	0
0.3	0	0.04	0.08	5.12	10.15	10.16	10.15	8.92	7.63	3.82	0
0.4	0	0.02	0.04	0.06	5.08	10.09	8.92	8.89	6.37	3.82	0
0.5	0	0	0	0	0	0	0	0	0	0	0
	y(0,t)=0										**y(1,t)=0**

Example: 2 Approximate the solution of the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, 0 < x < 1, t > 0,$

$u(0,t) = u(1,t) = 0, t > 0, \quad u(x,0) = \sin 2\pi x, 0 \le x \le 1 \;\; and \;\; \frac{\partial u}{\partial t}(x,0) = 0, \quad 0 \le x \le 1$

$with\ \Delta x = 0.25\ and\ \Delta t = 0.25$ for three time steps.

Solution:

Given $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \Rightarrow a^2 = 1 \Rightarrow a = 1.$

$$\Delta x = h = 0.25\ and\ \Delta t = k = 0.25$$

The explicit formula for solving wave equation is given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$u(x,0) = \sin 2\pi x$

$u(0.0,0) = \sin 2\pi(0) = 0 \qquad u(0.25,0) = \sin 2\pi(0.25) = 1$

$u(0.50,0) = \sin 2\pi(0.50) = 0 \qquad u(0.75,0) = \sin 2\pi(0.75) = -1$

$u(1.0,0) = \sin 2\pi(1) = 0$

x/t	0	0.25	0.5	0.75	1
0	0	1	0	−1	0
0.25	0	0	0	0	0
0.5	0	−1	0	1	0
0.75	0	0	0	0	0
1	0	1	0	−1	0
	$u(0,t) = 0$				$u(1,t) = 0$

Example: 3 Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $, 0 < x < 1, t > 0,\ u(0,t) = 0,\ u(1,t) = 0\ ,\ u(x,0) = x - x^2$ and $\frac{\partial u}{\partial t}(x,0) = 0$ taking h$= 0.2$ upto $t = 1$.

Solution:

Given $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow a^2 = 1 \ \Rightarrow a = 1.$

$$h = 0.2 \qquad \Rightarrow k = \frac{h}{a} = \frac{0.2}{1} = 0.2$$

The explicit formula for solving wave equation is given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$u(x,0) = x - x^2$

$$u(0{,}0) = 0 - 0^2 = 0$$

$$u(0.2{,}0) = 0.2 - 0.2^2 = 0.16$$

$$u(0.4{,}0) = 0.4 - 0.4^2 = 0.24$$

$$u(0.6{,}0) = 0.6 - 0.6^2 = 0.24$$

$$u(0.8{,}0) = 0.8 - 0.8^2 = 0.16$$

$$u(1.0{,}0) = 1 - 1^2 = 0$$

x/t	0	0.2	0.4	0.6	0.8
0	0	0.16	0.24	0.24	0.16
0.2	0	0.12	0.2	0.2	0.12
0.4	0	0.04	0.08	0.08	0.04
0.6	0	-0.04	-0.08	-0.08	-0.04
0.8	0	-0.12	-0.2	-0.2	-0.12
1	0	-0.16	-0.24	-0.24	-0.16
	$u(0,t) = 0$				$u(1,t) = 0$

Example: 4 Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $, 0 < x < 1, t > 0,\ u(0,t) = 0,\ u(1,t) = 0\ ,\ u(x,0) = 100(x - x^2)$ and $\frac{\partial u}{\partial t}(x,0) = 0$ taking h$= 0.2$ by using finite difference method for once time step.

Solution:

Given $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \Rightarrow a^2 = 1 \ \Rightarrow a = 1.$

$$h = 0.25 \qquad \Rightarrow k = \frac{h}{a} = \frac{0.25}{1} = 0.25$$

The explicit formula for solving wave equation is given by

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$u(x,0) = 100(x - x^2)$

$$u(0,0) = 100(0 - 0^2) = 0$$

$$u(0.25,0) = 100(0.25 - 0.25^2) = 18.75$$

$$u(0.50,0) = 100(0.5 - 0.5^2) = 25$$

$$u(0.75,0) = 100(0.75 - 0.75^2) = 18.75$$

$$u(1.0,0) = 100(1 - 1^2) = 0$$

x/t	**0**	**0.25**	**0.5**	**0.75**	**1**
0	0	18.75	25	18.75	0
0.25	0	12.5	18.75	12.5	0
0.5	0	0	0	0	0
	$u(0,t) = 0$				$u(1,t) = 0$

TWO DIMENSIONAL LAPLACE EQUATION

I. ELLIPTIC EQUATIONS

The Laplace equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

The standard five point formula is given by

$$u_i,j = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

The diagonal five point formula is given by

$$u_i,j = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$

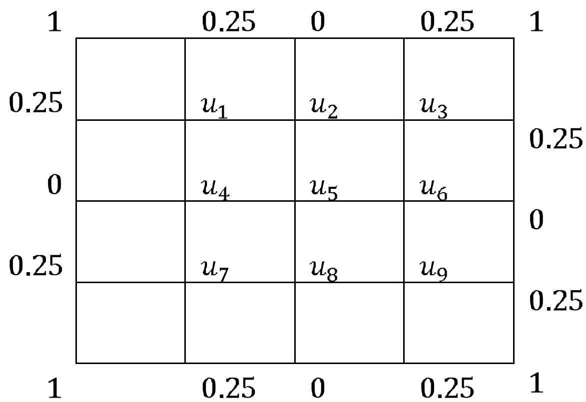
Example: 1

Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ in\ |x| < 1, |y| < 1\ with\ h = \frac{1}{2}\ and\ u(x, \pm 1) = x^2\ \&\ u(\pm 1, y) = y^2$

(i) $u(x,1) = x^2, -1 < x < +1$
(ii) $u(x,-1) = x^2, -1 < x < +1$

(iii) $u(1,y) = y^2, -1 < y < +1$
(iv) $u(-1,y) = y^2, -1 < y < +1$

Solution:



To find the rough values:

$$u_5 = \frac{1}{4}[0+0+0+0] = 0 \quad \{SFPF\}$$

$$u_1 = \frac{1}{4}[0+0+0+1] = 0.25 \quad \{DFPF\}$$

$$u_3 = \frac{1}{4}[1+0+0+0] = 0.25 \quad \{DFPF\}$$

$$u_7 = \frac{1}{4}[1+0+0+0] = 0.25 \quad \{DFPF\}$$

$$u_9 = \frac{1}{4}[1+0+0+0] = 0.25 \quad \{DFPF\}$$

To find the other rough values, we use **SFPF**

$$u_2 = \frac{1}{4}[0+0+0.25+0.25] = 0.125 \quad \{SFPF\}$$

$$u_4 = \frac{1}{4}[0+0+0.25+0.25] = 0\ .125 \quad \{SFPF\}$$

$$u_6 = \frac{1}{4}[0+0+0.25+0.25] = 0.125 \quad \{SFPF\}$$

$$u_8 = \frac{1}{4}[0+0+0.25+0.25] = 0.125 \quad \{SFPF\}$$

Here after we use only SFPF.

First Iteration:

$$u_1 = \frac{1}{4}[0.25 + 0.25 + 0.125 + 0.125] = 0.19$$

$$u_2 = \frac{1}{4}[0 + 0.19 + 0.25 + 0] = 0.11$$

$$u_3 = \frac{1}{4}[0.25 + 0.25 + 0.11 + 0.125] = 0.18$$

$$u_4 = \frac{1}{4}[0 + 0.19 + 0 + 0.25] = 0.11$$

$$u_5 = \frac{1}{4}[0.11 + 0.11 + 0.125 + 0.125] = 0.12$$

$$u_6 = \frac{1}{4}[0 + 0.12 + 0.18 + 0.25] = 0.14$$

$$u_7 = \frac{1}{4}[0.25 + 0.25 + 0.11 + 0.25] = 0.18$$

$$u_8 = \frac{1}{4}[0 + 0.12 + 0.18 + 0.25] = 0.14$$

$$u_9 = \frac{1}{4}[0.25 + 0.25 + 0.14 + 0.14] = 0.20$$

Second Iteration :

$u_1 = 0.19,\quad u_2 = 0.11,\quad u_3 = 0.18,\quad u_4 = 0.11,\quad u_5 = 0.12,\quad u_6 = 0.14,$
$u_7 = 0.18,\quad u_8 = 0.14,\quad u_9 = 0.20$

Third Iteration :

$u_1 = 0.18,\quad u_2 = 0.12,\quad u_3 = 0.19,\quad u_4 = 0.12,\quad u_5 = 0.13,\quad u_6 = 0.13,$
$u_7 = 0.19,\quad u_8 = 0.13,\quad u_9 = 0.19$

Poisson's Equation

An equation of the form $\nabla^2 u = f(x, y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \text{—}(1)$$

Is called Poisson's equation where $f(x, y)$ is a function of x & y.

The algorithm for solving Poisson's equation is

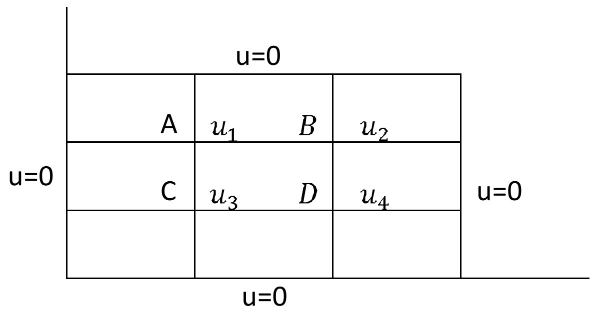
$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad --(2)$$

Example:

Solve $\nabla^2 u = -10(x^2 + y^2 + 10)$ over the square mesh with sides $x = 0,\ y = 0,\ x = 3,\ y = 3$ with $u = 0$ on the boundary and mesh length 1 unit.

Solution:

The boundary values at each side are equal, then $u_1 = u_4$. We need to find $u_1,\ u_2\ and\ u_3$.



The Partial differential equation is $\nabla^2 u = -10(x^2 + y^2 + 10),\ \ h = 1$

$$Now,\ u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh) \quad --(1)$$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = -10\,[i^2 + j^2 + 10] --(2)$$

Applying $(2)\ \ at\ \ the\ point\ \ A(i = 1, j = 2)$

$$0 + 0 + u_2 + u_3 - 4u_1 = -10[1 + 4 + 10]$$

$$-4u_1 + u_2 + u_3 = -150 \ \ \ldots\ldots(3)$$

Applying $(2)\ \ at\ \ the\ point\ \ B(i = 2, j = 2)$

$$u_1 + 0 + u_4 + 0 - 4u_2 = -10[4 + 4 + 10]$$

$$2u_1 - 4u_2 = -180 \ \ \ldots\ldots(4)$$

Applying $(2)\ \ at\ \ the\ point\ \ C(i = 1, j = 1)$

$$0 + u_4 + 0 + u_1 - 4u_3 = -10[1 + 1 + 10]$$

$$2u_1 - 4u_3 = -120 \ \ \ldots\ldots(5)$$

Solving the equations $(3),\ (4)\ and\ (5)$, we get

$$\boldsymbol{u_1 = u_4 = 75,\ u_2 = 82.5\ and\ u_3 = 67.5}$$

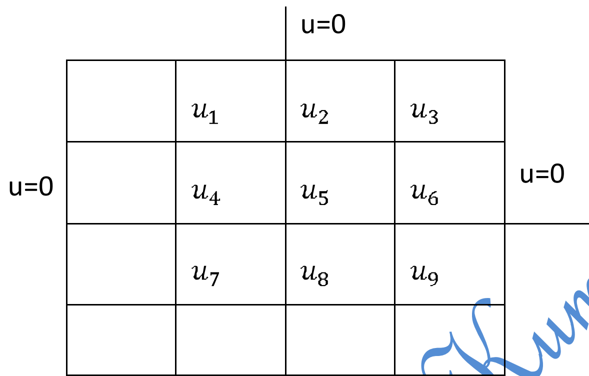
Example:

Solve $\nabla^2 u = 8x^2y^2$ for square mesh given $u = 0$ on the four boundaries dividing the square into 16 sub-squares of length 1 unit.

Solution: The values on the boundary are symmetric to each other.

$$\therefore \quad u_1 = u_3 = u_7 = u_9, \quad u_2 = u_4 = u_6 = u_8 \text{ and } u_5 \text{ not equal to any value}$$

So we need to find u_1, u_2 *and* u_5.



The Partial differential equation is $\nabla^2 u = 8x^2y^2, \ \ h = 1$

$$Now, \ u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 \, f(ih, jh) \quad --(1)$$

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = 8\, i^2 \, j^2 \quad --(2)$$

Applying (2) $at \ \ the\ point\ \ A(i = -1, j = -1)$

$$0 + 0 + u_2 + u_4 - 4u_1 = 8\,(-1)^2(-1)^2$$

$$-4u_1 + 2u_2 = 8 \quad \ldots\ldots(3)$$

Applying (2) $at \ \ the\ point\ \ B(i = 0, j = 1)$

$$u_1 + u_3 + u_5 - 4u_2 = 8(0)(1)$$

$$2u_1 - 4u_2 + u_5 = 0 \quad \ldots\ldots(4)$$

Applying (2) $at \ \ the\ point\ \ C(i = 0, j = 0)$

$$u_4 + u_6 + u_2 + u_8 - 4u_5 = 8(0)(0)$$

$$4u_2 - 4u_5 = 0 \quad \ldots\ldots(5)$$

Solving the equations (3), (4) *and* (5), we get

$$\boldsymbol{u_1 = u_3 = u_7 = u_9 = -3, \quad u_2 = u_4 = u_6 = u_8 = -2, \quad u_5 = -2}$$