UNIT-1

SOLUTIONS OF AN EQUATIONS & EIGEN VALUE PROBLES

NEWTONS METHOD OR NEWTON-RAPHSON METHOD

1. Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton- Raphson method.

Solution :

Let $f(x) = x^4 - x - 10 = 0$.

Now, $f(0) = (0)^4 - (0) - 10 = -10 \quad (-ve)$

$f(1) = (1)^4 - (1) - 10 = -10 \quad (-ve)$

$f(2) = (2)^4 - (2) - 10 = +4 \quad (+ve)$

Therefore the root lies between **1 & 2**.

Let us take $x_0 = 2$ $\{Near\ to\ zero\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots (1)$.

$$Let \quad f(x) = x^4 - x - 10 \text{ and } f^1(x) = 4x^3 - 1$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 2 - \left[\frac{f(2)}{f'(2)}\right].$$

$$x_1 = 2 - \left[\frac{(2)^4 - (2) - 10}{4(2)^3 - 1}\right] = 2 - \left[\frac{4}{31}\right].$$

$$x_1 = 1.8709 .$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 2 - \left[\frac{f(1.8709)}{f'(1.8709)}\right].$$

$$x_2 = 1.8709 - \left[\frac{(1.8709)^4 - (1.8709) - 10}{4(1.8709)^3 - 1}\right].$$

$$x_2 = 1.8709 - \left[\frac{0.3835}{25.199}\right].$$

$$x_2 = 1.856 .$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 1.856 - \left[\frac{f(1.856)}{f'(1.856)}\right].$$

$$x_3 = 1.856 - \left[\frac{(1.856)^4 - (1.856) - 10}{4(1.856)^3 - 1}\right].$$

$$x_3 = 1.856 - \left[\frac{0.010}{24.574}\right].$$

$$x_3 = 1.856 .$$

The root of the equation $x^4 - x = 10$ is 1.856.

2. **Using Newton's iterative method to find the root between 0 and 1 of $x^3 = 6x - 4$ correct to three decimal places.**

Solution :

Let $f(x) = x^3 - 6x + 4 \quad = 0.$

Now, $f(0) = (0)^3 - 6(0) + 4 = +4 \quad (+ve)$

$f(1) = (1)^3 - 6(1) + 4 = -1 \quad (-ve)$

Therefore the root lies between **0 & 1**.

Let us take $x_0 = 1 \quad \{Near\ to\ zero\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots (1)$.

$$Let \quad f(x) = x^3 - 6x + 4 \text{ and } f^1(x) = 3x^2 - 6$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 1 - \left[\frac{f(1)}{f'(1)}\right].$$

$$x_1 = 1 - \left[\frac{(1)^3 - 6(1) + 4}{3(1)^2 - 6}\right] = 1 - \left[\frac{-1}{-3}\right]$$

$$\boldsymbol{x_1 = 0.666}.$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.666 - \left[\frac{f(0.666)}{f'(0.666)}\right].$$

$$x_2 = 0.666 - \left[\frac{(0.666)^3 - 6(0.666) + 4}{3(0.666)^2 - 6}\right] = 0.666 - \left[\frac{0.28}{-4.65}\right].$$

$$\boldsymbol{x_2 = 0.73}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 0.73 - \left[\frac{f(0.73)}{f'(0.73)}\right].$$

$$x_3 = 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3(0.73)^2 - 6}\right] = 0.73 - \left[\frac{0.009}{-4.4013}\right].$$

$$\boldsymbol{x_3 = 0.7320}.$$

The root of the equation $\boldsymbol{x^3 - 6x + 4 = 0}$ **is 0.732**.

3. **Find the positive root of $3x - \cos x - 1 = 0$ correct to six decimal places by Newton method.**

Solution :

Let $f(x) = 3x - \cos x - 1 = 0.$

Now, $f(0) = 3(0) - \cos(0) - 1 = -2 \qquad (-ve)$

$f(1) = 3(1) - \cos(1) - 1 = 1.459698 \quad (-ve)$

Therefore the root lies between 0 & 1.

Let us take $\boldsymbol{x_0 = 1} \quad \{\boldsymbol{Near\ to\ zero}\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots (1)$.

$$Let \quad f(x) = 3x - \cos x - 1 \quad \text{and} \quad f^1(x) = 3 + \sin x$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 1 - \left[\frac{f(1)}{f'(1)}\right].$$

$$x_1 = 1 - \left[\frac{3(1) - \cos(1) - 1}{3 + \sin(1)}\right].$$

$$\boldsymbol{x_1 = 0.62002.}$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.62002 - \left[\frac{f(0.62002)}{f'(0.62002)}\right].$$

$$x_2 = 0.62002 - \left[\frac{3(0.62002) - \cos(0.62002) - 1}{3 + sin\,(0.62002)}\right].$$

$$\boldsymbol{x_2 = 0.60712\,.}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 0.60712 - \left[\frac{f(0.60712)}{f'(0.60712)}\right].$$

$$x_3 = 0.60712 - \left[\frac{3(0.60712) - \cos(0.60712) - 1}{3 + sin\,(0.60712)}\right].$$

$$\boldsymbol{x_3 = 0.6071\,.}$$

The root of the equation $\boldsymbol{x^4 - x = 10}$ **is** $\boldsymbol{0.60712}$.

4. Using Newton's iterative method solve $\boldsymbol{x \log_{10} x = 12.34}$ start with $\boldsymbol{x_0 = 10}$.

Solution :

Let $f(x) = x \log_{10} x - 12.34 \;\; = 0.$

Now, $f(0) = (0)^3 - 6(0) + 4 = +4 \quad (+ve)$

$f(1) = (1)^3 - 6(1) + 4 = -1 \quad (-ve)$

Therefore the root lies between 0 & 1.

Let us take $x_0 = 1 \;\; \{Near\ to\ zero\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots (1)\,.$

$$Let \quad f(x) = x^3 - 6x + 4 \quad \text{and} \quad f^1(x) = 3\,x^2 - 6$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 1 - \left[\frac{f(1)}{f'(1)}\right].$$

$$x_1 = 1 - \left[\frac{(1)^3 - 6(1) + 4}{3\,(1)^2 - 6}\right] = 1 - \left[\frac{-1}{-3}\right].$$

$$\boldsymbol{x_1 = 0.666\,.}$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.666 - \left[\frac{f(0.666)}{f'(0.666)}\right].$$

$$x_2 = 0.666 - \left[\frac{(0.666)^3 - 6(0.666) + 4}{3\,(0.666)^2 - 6}\right] = 0.666 - \left[\frac{0.28}{-4.65}\right].$$

$$\boldsymbol{x_2 = 0.73\,.}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 0.73 - \left[\frac{f(0.73)}{f'(0.73)}\right].$$

$$x_3 = 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3\,(0.73)^2 - 6}\right] = 0.73 - \left[\frac{0.009}{-4.4013}\right].$$

$$\boldsymbol{x_3 = 0.7320}\,.$$

The root of the equation $\boldsymbol{x^3 - 6x + 4 = 0}$ **is 0.732**.

5. **Find the positive root of $x^3 - 2x - 5 = 0$ Newton- Raphson –method.**

Solution :

Let $f(x) = x^3 - 2x - 5 = 0.$

Now, $f(0) = (0)^3 - 2(0) - 5 = -5 \quad (-ve)$

$f(1) = (1)^3 - 2(1) - 5 = -6 \quad (-ve)$

$f(2) = (2)^3 - 2(2) - 5 = -1 \quad (-ve)$

$f(3) = (3)^3 - 2(3) - 5 = +16 \quad (+ve)$

Therefore the root lies between **2 & 3.**

Let us take $x_0 = 2$ $\{Near\ to\ zero\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots(1)$.

$$Let \qquad f(x) = x^3 - 6x + 4 \text{ and } f^1(x) = 3\,x^2 - 6.$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 2 - \left[\frac{f(2)}{f'(2)}\right].$$

$$x_1 = 2 - \left[\frac{(2)^3 - 2(2) - 5}{3\,(2)^2 - 6}\right] = 2 - \left[\frac{-1}{10}\right] = \mathbf{2.1}$$

$$x_2 = 2.1 - \left[\frac{f(2.1)}{f'(2.1)}\right] = 1.8709 - \left[\frac{0.061}{11.23}\right] = \mathbf{2.0946}$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 2.0946 - \left[\frac{f(2.0946)}{f'(2.0946)}\right] = \mathbf{2.0946}$$

The root of the equation $\boldsymbol{x^3 - 6x + 4 = 0}$ **is 2.0946**.

6. **Find the positive root of $\cos x = x\,e^x$ by Newton- Raphson –method. Take $x_0 = 0.5$.**

Solution :

Let $f(x) = \cos x - x\,e^x = 0.$

Given $x_0 = 0.5$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots(1)$.

$$Let \qquad f(x) = \cos x - x\,e^x \text{ and}$$

$$f^1(x) = -\sin x - xe^x - e^x \implies f^1(x) = -\sin x - (x+1)e^x$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 0.5 - \left[\frac{f(0.5)}{f'(0.5)}\right].$$

$$x_1 = 0.5 - \left[\frac{\cos(0.5) - 0.5\ e^{0.5}}{-\sin(0.5) - (0.5+1)e^{0.5}}\right] = 0.5 - \left[\frac{0.0532}{-2.9525}\right].$$

$$\boldsymbol{x_1 = 0.5180}\,.$$

$$\boldsymbol{x_2 = 0.5178}\,.$$

$$\boldsymbol{x_3 = 0.5178}\,.$$

The root of the equation $\boldsymbol{\cos x = x\, e^x}$ **is 0.5178**.

7. **Using Newton's iterative method to find the negative root of $\boldsymbol{x^2 + 4\sin x = 0}$.**

Solution :

Let $f(x) = x^2 + 4\sin x = 0$.

Now, $f(0) = 0^2 + 4\sin(0) = +0 \quad (+ve)$

$f(1) = 1^2 + 4\sin(1) = +4.3659 \quad (+ve)$

$f(2) = 2^2 + 4\sin(2) = +7.6372 \quad (+ve)$

$f(-1) = -1^2 + 4\sin(-1) = - -2.3659 \quad (-ve)$

$f(-2) = -2^2 + 4\sin(-2) = +0.3628 \quad (+ve)$

Therefore the root lies between $\boldsymbol{-1}$ & $\boldsymbol{-2}$.

Let us take $\boldsymbol{x_0 = -2}$ $\{\boldsymbol{Near\ to\ zero}\}$.

The Newton- Raphson formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \ldots (1)$.

$$Let \qquad f(x) = x^2 + 4\sin x \ \text{ and } \ f^1(x) = 2\,x + 4\cos x$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = -2 - \left[\frac{f(-2)}{f'(-2)}\right].$$

$$x_1 = 1 - \left[\frac{(-2)^2 + 4\sin(-2)}{2(-2) + 4\cos(-2)}\right] = -2 - \left[\frac{0.3628}{-5.6646}\right].$$

$$\boldsymbol{x_1 = -1.9359}\,.$$

$$\boldsymbol{x_2 = -1.9338}\,.$$

$$\boldsymbol{x_3 = -1.9338}\,.$$

The root of the equation $\boldsymbol{x^2 + 4\sin x = 0}$ **is** $\boldsymbol{-1.9338}$.

FIXED POINT ITERATION OR ITERATION METHOD

The condition for convergence of a method

Let $f(x) = 0$ be the given equation whose actual root is r. The equation $f(x) = 0$ be written as $x = g(x)$. Let I be the interval containing the root $x = r$. If $|g'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, \ldots \ldots . x_n$ will converge to r, if the initial starting value x_0 is chosen in I.

Note 1. Since $|x_n - r| \le K|x_{n-1} - r|$ where K is a constant the convergence is linear and the convergence is of order 1.

Note 2. The sufficient condition for the convergence is $|g'(x)| < 1$ for all x in I

1. **Find the positive root of $x^2 - 2x - 3 = 0$ by Iteration method.**

   Solution :

   Let $f(x) = x^2 - 2x - 3 = 0$.

   Now, $f(0) = (0)^2 - 2(0) - 3 = -10 \quad (-ve)$

   $f(1) = (0)^2 - 2(0) - 3 = -10 \quad (-ve)$

   $f(2) = (0)^2 - 2(0) - 3 = +4 \quad (+ve)$

   Therefore the root lies between 1 & 2.

   Let us take $x_0 = 2$ $\{Near\ to\ zero\}$.

$$x^2 - 2x - 3 = 0 \Rightarrow x^2 = 2x + 3$$

$$\Rightarrow x = \sqrt{2x + 3}$$

$$\Rightarrow x = g(x) = \sqrt{2x + 3}$$

   Let $x_0 = 2$

$$x_1 = g(x_0) = \sqrt{2x_0 + 3} = \sqrt{2(2) + 3} = 2.6457$$

$$x_2 = g(x_1) = \sqrt{2x_1 + 3} = \sqrt{2(2.6457) + 3} = 2.8795$$

$$x_3 = g(x_2) = \sqrt{2x_2 + 3} = \sqrt{2(2.8795) + 3} = 2.9595$$

$$x_4 = g(x_3) = \sqrt{2x_3 + 3} = \sqrt{2(2.9595) + 3} = 2.9864$$

$$x_5 = g(x_4) = \sqrt{2x_4 + 3} = \sqrt{2(2.9864) + 3} = 2.99549$$

$$x_6 = g(x_5) = \sqrt{2x_5 + 3} = \sqrt{2(2.99549) + 3} = 2.9985$$

$$x_7 = g(x_6) = \sqrt{2x_6 + 3} = \sqrt{2(2.9985) + 3} = 2.9995$$

$$x_8 = g(x_7) = \sqrt{2x_7 + 3} = \sqrt{2(2.9995) + 3} = 2.9998$$

$$x_9 = g(x_8) = \sqrt{2x_8 + 3} = \sqrt{2(2.9998) + 3} = 2.9999$$

$$x_{10} = g(x_9) = \sqrt{2x_9 + 3} = \sqrt{2(2.9999) + 3} = 2.9999$$

   Hence the root of the equation is $\boldsymbol{x^2 - 2x - 3 = 0}$ is $\mathbf{2.9999}$.

2. **Find the Real root of the equation $x^3 + x^2 - 100$ by Fixed point iteration method.**

   Solution:

$$\text{Let} \quad f(x) = x^3 + x^2 - 100 = 0.$$

$$f(0) = (0)^3 + (0)^2 - 100 = -100 \quad (-ve).$$

$$f(1) = (1)^3 + (1)^2 - 100 = -98 \quad (-ve).$$

$$f(2) = (2)^3 + (2)^2 - 100 = -88 \quad (-ve).$$
$$f(3) = (3)^3 + (3)^2 - 100 = -64 \quad (-ve).$$
$$f(4) = (4)^3 + (4)^2 - 100 = -20 \quad (-ve).$$
$$f(5) = (5)^3 + (5)^2 - 100 = +50 \quad (+ve).$$

The root lies between 4 & 5.

$$Since \quad x^3 + x^2 - 100 = 0$$
$$\Rightarrow \quad x^2(x+1) = 100$$
$$\Rightarrow \quad x^2 = \frac{100}{(x+1)}$$
$$\Rightarrow \quad x = g(x) = \frac{10}{\sqrt{x+1}} = 10\,[x+1]^{\frac{1}{2}}$$

Now, $\quad g'(x) = 10\left(\frac{1}{2}\right)[x+1]^{\left(-\frac{3}{2}\right)} = 5\,[x+1]^{\left(-\frac{3}{2}\right)}$

$$g'(4) = 5\,[4+1]^{\left(-\frac{3}{2}\right)} < 1$$
$$g'(5) = 5\,[5+1]^{\left(-\frac{3}{2}\right)} < 1$$

So that we can use the iteration method.

Let $x_0 = 4$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4\ +1}} = \frac{10}{2.236} = 4.4721$$
$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1+1}} = \frac{10}{\sqrt{4.4721\ +1}} = \frac{10}{2.1147} = 4.2748$$
$$x_3 = g(x_2) = \frac{10}{\sqrt{x_2+1}} = \frac{10}{\sqrt{4.2748+1}} = 4.3541$$
$$x_4 = g(x_3) = \frac{10}{\sqrt{x_3+1}} = \frac{10}{\sqrt{4.3541+1}} = 4.3217$$
$$x_5 = g(x_4) = \frac{10}{\sqrt{x_4+1}} = \frac{10}{\sqrt{4.3217+1}} = 4.3348$$
$$x_6 = g(x_5) = \frac{10}{\sqrt{x_5+1}} = \frac{10}{\sqrt{4.3348+1}} = 4.3295$$
$$x_7 = g(x_6) = \frac{10}{\sqrt{x_6+1}} = \frac{10}{\sqrt{4.3295\ +1}} = 4.3316$$
$$x_8 = g(x_7) = \frac{10}{\sqrt{x_7+1}} = \frac{10}{\sqrt{4.3316\ +1}} = 4.3307$$
$$x_9 = g(x_8) = \frac{10}{\sqrt{x_8+1}} = \frac{10}{\sqrt{4.3307+1}} = 4.3311$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{x_9 + 1}} = \frac{10}{\sqrt{4.3311 + 1}} = 4.3310$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{x_{10} + 1}} = \frac{10}{\sqrt{4.3310 + 1}} = 4.3310$$

Hence the root of the equation is $\boldsymbol{x^3 + x^2 - 100 = 0}$ **is** $\mathbf{4.3310}$.

3. **Find the real root of the equation** $\cos x = 3x - 1$ **correct to five decimal places using fixed point iteration method.**

Solution:

$$\text{Let} \quad f(x) = \cos x - 3x + 1 = 0.$$

$$f(0) = \cos(0) - 3(0) + 1 = 2 \qquad (+ve).$$

$$f(1) = \cos(1) - 3(1) + 1 = -1.4597 \quad (+ve).$$

The root lies between 0 & 1.

$$Since \qquad \cos x - 3x + 1 = 0$$

$$\Rightarrow \; 3x = \cos x + 1$$

$$\Rightarrow \quad x = \frac{1}{3}(1 + \cos x)$$

$$\Rightarrow \quad x = g(x) = \frac{1}{3}(1 + \cos x)$$

Now, $g'(x) = \frac{1}{3}(-\sin x) = -\frac{1}{3}\sin x$

$$g'(0) = -\frac{1}{3}\sin(0) = \;\; 0 \quad < 1$$

$$g'(1) = -\frac{1}{3}\sin(1) = \;\; 0.284 < 1$$

So that we can use the iteration method.

Let $x_0 = 4$

$$x_1 = g(x_0) = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + (-0.6536)\;) = 0.11545$$

$$x_2 = g(x_1) = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos(0.11545)\;\;) = 0.6644$$

$$x_3 = g(x_2) = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos[0.6644]\;\;) = 0.5957$$

$$x_4 = g(x_3) = \frac{1}{3}(1 + \cos x_3) = 0.6092$$

$$x_5 = g(x_4) = \frac{1}{3}(1 + \cos x_4) = 0.60669$$

$$x_6 = g(x_5) = \frac{1}{3}(1 + \cos x_5) = 0.60717$$

$$x_7 = g(x_6) = \frac{1}{3}(1 + \cos x_6) = 0.60708$$

$$x_8 = g(x_7) = \frac{1}{3}(1 + \cos x_7) = 0.60710$$

$$x_9 = g(x_8) = \frac{1}{3}(1 + \cos x_8) = 0.60710$$

Hence the root of the equation is $\boldsymbol{\cos x = 3x - 1}$ **is** $\mathbf{0.60710}$.

4. **Solve by iteration method** $\boldsymbol{e^x - 3x = 0}$

Solution :

$$\text{Let} \quad f(x) = e^x - 3x = 0.$$

$$f(0) = e^0 - 3(0) = 1 \qquad (+ve).$$

$$f(1) = e^1 - 3(1) = - \qquad (+ve).$$

The root lies between 0 & 1.

$$Since \qquad e^x - 3x = 0$$

$$\Rightarrow \quad 3x = e^x \quad \Rightarrow \quad x = \frac{1}{3}(e^x)$$

$$\Rightarrow \quad x = g(x) = \frac{1}{3}(e^x)$$

Now, $|g'(x)| = \frac{1}{3}(e^x)$

$$|g'(0)| = \frac{1}{3}e^0 = \frac{1}{3} \quad < 1$$

$$|g'(1)| = \frac{1}{3}e^1 = \frac{e}{3} \quad < 1$$

So that we can use the iteration method.

Let $x_0 = 0$

$$x_1 = g(x_0) = \frac{1}{3}e^{x_0} = \frac{1}{3}(e^0) = 0.3334$$

$$x_2 = g(x_1) = \frac{1}{3}e^{x_1} = \frac{1}{3}(e^{0.3334}) = 0.4652$$

$$x_3 = g(x_2) = \frac{1}{3}e^{x_2} = 0.5308$$

$$x_{14} = g(x_9) = \frac{1}{3}e^{x_{13}} = 0.6186$$

Hence the root of the equation is $\boldsymbol{e^x - 3x = 0}$ **is** $\mathbf{0.618}$.

GAUSS ELIMINATION AND GAUSS JORDAN METHOD

1. **Solve the system of equations by (i) Gauss elimination method (ii) Gauss Jordan method.**
$\boldsymbol{2x + 4y + 8z = 41, \;\; 4x + 6y + 10\,z = 56, \;\; 6x + 8y + 10z = 64\,.}$

Solution : (i). **Gauss elimination method** :

Let the given system of equations be $2x + 4y + 8z = 41$

$$4x + 6y + 10\,z = 56$$

$$6x + 8y + 10z = 64$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 10 \\ 6 & 8 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 41 \\ 56 \\ 64 \end{bmatrix}$$

$$Here\ \ [A,B] = \begin{bmatrix} 2 & 4 & 8 & 41 \\ 4 & 6 & 10 & 56 \\ 6 & 8 & 10 & 64 \end{bmatrix}$$

Now, we need to make A as an upper triangular matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ and\ third\ row\ by\ using\ first\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 4 & 8 & 41 \\ 0 & -2 & -6 & -26 \\ 0 & 4 & -14 & -59 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow R_2 - 2\,R_1 \\ R_3 \Leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 4 & 8 & 41 \\ 0 & -2 & -6 & -26 \\ 0 & 0 & -2 & -7 \end{bmatrix} \quad R_3 \Leftrightarrow R_3 - 2\,R_2$$

This is an upper triangular matrix. From the above matrix we have

$$-2\,z = -7 \ \Rightarrow\ z = \frac{7}{2} = 3.5$$

$$-2\,y - 6z = -26 \ \Rightarrow\ -2\ y - 6\left(\frac{7}{2}\right) = -26$$

$$\Rightarrow\ -2\,y = -26 + 21 \ \Rightarrow\ -2\,y = -5$$

$$\Rightarrow\ \ y = \frac{5}{2} = 2.5$$

$$2x + 4y + 8z = 41$$

$$\Rightarrow 2x + 4\left(\frac{5}{2}\right) + 8\left(\frac{7}{2}\right) = 41 \ \Rightarrow 2x = 41 - 10 - 28$$

$$\Rightarrow 2x = 3 \ \ \Rightarrow x = \frac{3}{2} = 1.5$$

Hence the solution is $\boldsymbol{x = 1.5\,,\ \ y = 2.5\ \ \ and\ \ \ z = 3.5}$

(ii) **Gauss Jordan method:** Let the given system of equations be

$$2x + 4y + 8z = 41$$

$$4x + 6y + 10\,z = 56$$

$$6x + 8y + 10z = 64$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 4 & 8 \\ 4 & 6 & 10 \\ 6 & 8 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 41 \\ 56 \\ 64 \end{bmatrix}$$

$$Here\ \ [A,B] = \begin{bmatrix} 2 & 4 & 8 & 41 \\ 4 & 6 & 10 & 56 \\ 6 & 8 & 10 & 64 \end{bmatrix}$$

Now, we need to make A as a diagonal matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ and\ third\ row\ by\ using\ first\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 4 & 8 & 41 \\ 0 & -2 & -6 & -26 \\ 0 & 4 & -14 & -59 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow R_2 - 2\,R_1 \\ R_3 \Leftrightarrow R_3 - 3R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 4 & 8 & 41 \\ 0 & -2 & -6 & -26 \\ 0 & 0 & -2 & -7 \end{bmatrix} \quad R_3 \Leftrightarrow R_3 - 2\,R_2$$

$$Fix\ the\ third\ \ row, change\ first\ \ and\ second\ \ row\ by\ using\ third\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 4 & 0 & 13 \\ 0 & -2 & 0 & -5 \\ 0 & 0 & -2 & -7 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow R_1 + 4\,R_3 \\ R_2 \Leftrightarrow R_2 - 3\,R_3 \end{matrix}$$

$$Fix\ the\ second\ \ \&\ \ third\ \ row, change\ first\ \ \ by\ using\ second\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 0 & 0 & 3 \\ 0 & -2 & 0 & -5 \\ 0 & 0 & -2 & -7 \end{bmatrix} \quad R_1 \Leftrightarrow \ R_1 + 2\,R_2$$

Which is a diagonal matrix, from the matrix, we have

$$2\,x = 3 \quad \Rightarrow \quad x = \frac{3}{2} = 1.5$$

$$-2\,y = -5 \quad \Rightarrow \quad y = \frac{5}{2} = 2.5$$

$$-\,2\,z = -7 \quad \Rightarrow \quad z = \frac{7}{2} = 3.5$$

Hence the solution is $\boldsymbol{x = 1.5\,,\ \ y = 2.5\ \ \ and\ \ \ z = 3.5}$

2. **Solve the system of equations by (I) Gauss elimination method (ii) Gauss Jordan method.**

   $\boldsymbol{2x + 3y - z = 5,\ \ 4x + 4y - 3\,z = 3,\ \ \ 2\,x - 3\,y + 2\,z = 2.}$

Solution :

(i**). Gauss elimination method**:

Let the given system of equations be

$$2x + 3y - z = 5$$

$$4x + 4y - 3\,z = 3$$

$$2\,x - 3\,y + 2\,z = 2$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$Here\ \ [A,B] = \begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{bmatrix}$$

Now, we need to make A as a upper triangular matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ and\ third\ row\ by\ using\ first\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow R_2 - 2\,R_1 \\ R_3 \Leftrightarrow R_3 - R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{bmatrix} \quad R_3 \Leftrightarrow R_3 - 3\,R_2$$

This is an upper triangular matrix. From the above matrix we have

$$6\,z = 18 \quad \Rightarrow \quad z = 3$$

$$-2\,y - \ z = \ -7 \quad \Rightarrow \ -2\ \ y - 3 = \ -7$$

$$\Rightarrow \ -2\,y = \ - -7 + 3 \ \Rightarrow \ -2\,y = \ -4$$

$$\Rightarrow \quad y = 2$$

$$2x + 3y - z = 5$$

$$\Rightarrow 2x + 3\,(2) - 3 = 5 \ \Rightarrow 2x = 5 - 6 + 3$$

$$\Rightarrow x = 1$$

Hence the solution is $\quad \boldsymbol{x = 1\,,\ \ y = 2\ \ and\ \ z = 3}$

(ii) **Gauss Jordan method**:

Let the given system of equations be $\quad 2x + 3y - z = 5$

$$4x + 4y - 3\,z = 3$$

$$2\,x - 3\,y + 2\,z = 2$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$$

$$Here\ \ [A,B] = \begin{bmatrix} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{bmatrix}$$

Now, we need to make A as a diagonal matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ and\ third\ row\ by\ using\ first\ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow R_2 - 2\,R_1 \\ R_3 \Leftrightarrow R_3 - R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A,B] \sim \begin{bmatrix} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{bmatrix} \quad R_3 \Leftrightarrow R_3 - 3\,R_2$$

$$Fix\ the\ third\ row, change\ first\ and\ second\ row\ by\ using\ third\ row.$$

$$[A,B] \sim \begin{bmatrix} 12 & 18 & 0 & 48 \\ 0 & -12 & 0 & -24 \\ 0 & 0 & 6 & 18 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow 6\,R_1 + R_3 \\ R_2 \Leftrightarrow 6\,R_2 + R_3 \end{matrix}$$

$$Fix\ the\ second\ \&\ third\ row, change\ first\ by\ using\ second\ row.$$

$$[A,B] \sim \begin{bmatrix} 144 & 0 & 0 & 144 \\ 0 & -12 & 0 & -24 \\ 0 & 0 & 6 & 18 \end{bmatrix} \quad R_1 \Leftrightarrow 12\,R_1 + 18\,R_2$$

Which is a diagonal matrix, from the matrix, we have

$$144\ x = 144 \quad \Rightarrow \quad x = 1$$

$$-12\,y = -24 \quad \Rightarrow \quad y = 2$$

$$6\,z = 18 \quad \Rightarrow \quad z = 3$$

Hence the solution is $\boldsymbol{x = 1\ ,\ \ y = 2\ \ and\ \ z = 3}$

3. Solve the system of equations by (i) Gauss elimination method (ii) Gauss Jordan method.

$\boldsymbol{10x - 2y + 3z = 23,\ \ 2x + 10y - 5z = -33,\ \ 3x - 4y + 10z = 41}$ **.**

Solution:

(i). **Gauss elimination method:**

Let the given system of equations be $10x - 2y + 3z = 23$

$$2x + 10y - 5z = -33$$

$$3x - 4y + 10z = 41$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

$$Here\ \ [A,B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

Now, we need to make A as a upper triangular matrix.

$$Fix\ the\ first\ row, change\ second\ and\ third\ row\ by\ using\ first\ row.$$

$$[A,B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow 5R_2 - R_1 \\ R_3 \Leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ row, change\ the\ third\ row\ by\ using\ second\ row.$$

$$[A,B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3 \Leftrightarrow 52R_3 + 34R_2$$

This is an upper triangular matrix. From the above matrix we have

$$3780\,z = 11340 \quad \Rightarrow \quad z = 3$$

$$52\,y - 28\,z = -188 \quad \Rightarrow \quad 52\,y - 28\,(3) = -188$$

$$\Rightarrow\ 52\,y = -188 + 84 = 104$$

$$\Rightarrow\ y = -2$$

$$10x - 2(-2) + 3(3) = 23 \ \Rightarrow\ x = 1$$

Hence the solution is $x = 1, y = -2\ \ and\ z = 3$

(ii) **Gauss Jordan method:**

Let the given system of equations be $10x - 2y + 3z = 23$

$$2x + 10y - 5z = -33$$

$$3x - 4y + 10z = 41$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 10 & -2 & 3 \\ 2 & 10 & -5 \\ 3 & -4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 23 \\ -33 \\ 41 \end{bmatrix}$$

$$Here\ \ [A, B] = \begin{bmatrix} 10 & -2 & 3 & 23 \\ 2 & 10 & -5 & -33 \\ 3 & -4 & 10 & 41 \end{bmatrix}$$

Now, we need to make A as a diagonal matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ and\ third\ row\ by\ using\ first\ row.$$

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & -34 & 91 & 341 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow 5R_2 - R_1 \\ R_3 \Leftrightarrow 10R_3 - 3R_1 \end{matrix}$$

$$Fix\ the\ first\ \&\ second\ \ row, change\ the\ \ third\ row\ by\ using\ second\ \ row.$$

$$[A, B] \sim \begin{bmatrix} 10 & -2 & 3 & 23 \\ 0 & 52 & -28 & -188 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_3 \Leftrightarrow 52R_3 + 34R_2$$

$$Fix\ the\ third\ \ row, change\ first\ \ and\ second\ \ row\ by\ using\ third\ row.$$

$$[A, B] \sim \begin{bmatrix} 12600 & -2520 & 0 & 17640 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow 1260R_1 - R_3 \\ R_2 \Leftrightarrow 135R_2 + 3R_3 \end{matrix}$$

$$Fix\ the\ second\ \ \&\ \ third\ \ row, change\ first\ \ \ by\ using\ second\ row.$$

$$[A, B] \sim \begin{bmatrix} 88452000 & 0 & 0 & 88452000 \\ 0 & 7020 & 0 & -14040 \\ 0 & 0 & 3780 & 11340 \end{bmatrix} \quad R_1 \Leftrightarrow 7020\,R_1 + 2520\,R_2$$

Which is a diagonal matrix, from the matrix, we have

$$3780\,z = 11340 \ \Rightarrow\ z = 3$$

$$7020\,y = -14040 \ \Rightarrow\ y = -2$$

$$88452000\,x = 88452000 \ \Rightarrow\ x = 1$$

Hence the solution is $x = 1, y = -2\ \ and\ z = 3$

4. **Solve the system of equations by (i) Gauss elimination method (ii) Gauss Jordan method.**

   $\mathbf{2x + 3y = 3 \quad 7x - 3y = 4}$ **.**

Solution: (i) **Gauss elimination method**

Let the given system be $2x + 3y = 3$

$$7x - 3y = 4$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 3 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$Here\ [A,B] = \begin{bmatrix} 2 & 3 & 3 \\ 7 & -3 & 4 \end{bmatrix}$$

Now, we need to make A as an upper triangular matrix.

$$Fix\ the\ first\ row, change\ second\ \ by\ using\ first\ row.$$

$$[A,B] = \begin{bmatrix} 2 & 3 & 3 \\ 0 & -27 & -13 \end{bmatrix} \quad R_2 \Leftrightarrow 2\,R_2 - 7\,R_1$$

This is an upper triangular matrix. From the above matrix we have

$$-27\ z = -13 \quad \Rightarrow \quad z = \frac{13}{27} = 0.4814$$

$$2\,x + 3y = 3 \quad \Rightarrow \quad 2x + 3(0.4814) = 3$$

$$\Rightarrow \quad 2x = 3 - 3(0.4814) = 1.5556$$

$$\Rightarrow \quad x = \frac{1.5556}{2} = 0.77778$$

Hence the solution is $\boldsymbol{x = 0.7778,\ \ y = 0.4814}$

(i) . **Gauss – Jordan Method :**

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 2 & 3 \\ 7 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$Here\ [A,B] = \begin{bmatrix} 2 & 3 & 3 \\ 7 & -3 & 4 \end{bmatrix}$$

Now, we need to make A as a Diagonal triangular matrix.

$$Fix\ the\ first\ row, change\ second\ \ by\ using\ first\ row.$$

$$[A,B] = \begin{bmatrix} 2 & 3 & 3 \\ 0 & -27 & -13 \end{bmatrix} \quad R_2 \Leftrightarrow 2\,R_2 - 7\,R_1$$

$$Fix\ the\ Second\ \ row, change\ first\ \ by\ using\ second\ row.$$

$$[A,B] = \begin{bmatrix} 54 & 0 & 42 \\ 0 & -27 & -13 \end{bmatrix} \quad R_1 \Leftrightarrow 27\,R_1 + 3\,R_2$$

which is a diagonal matrix, from the matrix we have

$$54\,x = 42 \quad \Rightarrow \quad x = 0.7778$$

$$-27\,z = -13 \quad \Rightarrow \quad z = 0.4814$$

5. Solve the system of equations by (i) Gauss elimination method (ii) Gauss Jordan method.

$\boldsymbol{11x + 3y = 17\,,\quad 2x + 7y = 16\,.}$

Solution : (i) Gauss elimination method

Let the given system be $11x + 3y = 17$

$$2x + 7y = 16$$

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 11 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \end{bmatrix}$$

$$Here\ \ [A, B] = \begin{bmatrix} 11 & 3 & 17 \\ 2 & 7 & 16 \end{bmatrix}$$

Now, we need to make A as an upper triangular matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ \ by\ using\ first\ row.$$

$$[A, B] = \begin{bmatrix} 11 & 3 & 17 \\ 0 & 71 & 142 \end{bmatrix} \qquad R_2 \Leftrightarrow 11\,R_2 - 2\,R_1$$

This is an upper triangular matrix. From the above matrix we have

$$71\ z = 142 \quad \Longrightarrow \quad z = \frac{142}{71} = 2$$

$$11\,x + 3y = 17 \quad \Longrightarrow \quad 11\,x + 3(2) = 17$$

$$\Longrightarrow \quad 11\,x = 17 - 6 = 11$$

$$\Longrightarrow \quad x = 1$$

Hence the solution is $\boldsymbol{x = 1,\ \ y = 2}$

(ii). Gauss – Jordan Method :

The given system is equivalent to $A\,X = B$

$$\begin{bmatrix} 11 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 17 \\ 16 \end{bmatrix}$$

$$Here\ \ [A, B] = \begin{bmatrix} 11 & 3 & 17 \\ 2 & 7 & 16 \end{bmatrix}$$

Now, we need to make A as a Diagonal triangular matrix.

$$Fix\ the\ first\ \ row, change\ second\ \ \ by\ using\ first\ row.$$

$$[A, B] = \begin{bmatrix} 11 & 3 & 17 \\ 0 & 71 & 142 \end{bmatrix} \qquad R_2 \Leftrightarrow 11\,R_2 - 2\,R_1$$

$$Fix\ the\ Second\ \ row, change\ first\ \ \ by\ using\ second\ row.$$

$$[A, B] = \begin{bmatrix} 781 & 0 & 781 \\ 0 & 71 & 142 \end{bmatrix} \qquad R_1 \Leftrightarrow 71\,R_1 - 3\,R_2$$

Which is a diagonal matrix, from the matrix we have

$$781\,x = 781 \quad \Longrightarrow \quad x = 1$$

$$71\,y = 142 \quad \Longrightarrow \quad y = 2$$

Hence the solution is $\boldsymbol{x = 1,\ \ y = 2}$

ITERATIVE METHODS

Gauss Jacobi and Gauss Siedal Method of Iteration

Consider the system of equations, $a_1x + b_1y + c_1z = d_1$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3 \quad \dots(1)$$

If the given system of equations obeys the condition, we can use Gauss Jacobi or Gauss Siedal Iteration methods.

$$|a_1| > |b_1| + |c_1| \;, \quad |b_2| > |a_2| + |c_2| \;, \quad |c_3| > |a_3| + |b_3|$$

Gauss Jacobi Method : The general n^{th} order iteration is

$$x^{(n+1)} = \frac{1}{a_1}\left(d_1 - b_1\, y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(n+1)} = \frac{1}{b_2}\left(d_2 - a_2\, x^{(r)} - c_2\, z^{(r)}\right)$$

$$z^{(n+1)} = \frac{1}{c_3}\left(d_3 - a_3\, x^{(r)} - b_3\, y^{(r)}\right) \quad \dots(2)$$

Gauss –Siedal Method :

$$x^{(n+1)} = \frac{1}{a_1}\left(d_1 - b_1\, y^{(r)} - c_1 z^{(r)}\right)$$

$$y^{(n+1)} = \frac{1}{b_2}\left(d_2 - a_2\, x^{(r+1)} - c_2\, z^{(r)}\right)$$

$$z^{(n+1)} = \frac{1}{c_3}\left(d_3 - a_3\, x^{(r+1)} - b_3\, y^{(r+1)}\right) \quad \dots(3)$$

1. **Solve the following system of equations by Gauss-Jacobi and Gauss-Siedal method of Iteration.**

   $\boldsymbol{27x + 6y - z = 85,\;\; x + y + 54z = 110,\;\; 6x + 15y + 2z = 72.}$

   Solution : As the coefficient matrix is not diagonally dominant in the coefficient matrix we rearrange the equations,

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Since, $|27| > |6| + |1|$, $|15| > |6| + |2|$, $|54| > |1| + |1|$

So that we can use Gauss iterative method,

Since the diagonal elements are dominant in the coefficient matrix, we rewrite $\boldsymbol{x, y, z}$ as follows

$$x = \frac{1}{27}(85 - 6y + z)$$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

$$z = \frac{1}{54}(110 - x - y)$$

Gauss Jacobi Method : Let the initial values be $\boldsymbol{x = 0, y = 0, z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{27}[85 - 6(0) + (0)] = \frac{1}{27}[85] = 3.148$$

$$y^{(1)} = \frac{1}{15}[72 - 6(0) - 2(0)] = \frac{1}{15}[72] = 4.8$$

$$z^{(1)} = \frac{1}{54}[110 - (0) - (0)] = \frac{1}{54}[110] = 2.037$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{27}\left(85 - 6y^{(1)} + z^{(1)}\right) = \frac{1}{27}[85 - 6(4.8) + (2.037)] = 2.157$$

$$y^{(2)} = \frac{1}{15}\left(72 - 6x^{(1)} - 2z^{(1)}\right) = \frac{1}{15}[72 - 6(3.148) - 2(2.037)] = 3.269$$

$$z^{(2)} = \frac{1}{54}\left(110 - x^{(1)} - y^{(1)}\right) = \frac{1}{54}[110 - (3.148) - (4.8)] = 1.890$$

3^{rd} Iteration :

$$x^{(3)} = \frac{1}{27}\left(85 - 6y^{(2)} + z^{(2)}\right) = \frac{1}{27}[85 - 6(3.269) + (1.890)] = 2.492$$

$$y^{(3)} = \frac{1}{15}\left(72 - 6x^{(2)} - 2z^{(2)}\right) = \frac{1}{15}[72 - 6(2.157) - 2(1.890)] = 3.685$$

$$z^{(3)} = \frac{1}{54}\left(110 - x^{(2)} - y^{(2)}\right) = \frac{1}{54}[110 - (2.157) - (3.269)] = 1.937$$

4^{th} Iteration :

$$x^{(4)} = \frac{1}{27}\left(85 - 6y^{(3)} + z^{(3)}\right) = \frac{1}{27}[85 - 6(3.685) + (1.937)] = 2.401$$

$$y^{(4)} = \frac{1}{15}\left(72 - 6x^{(3)} - 2z^{(3)}\right) = \frac{1}{15}[72 - 6(2.492) - 2(1.937)] = 3.545$$

$$z^{(4)} = \frac{1}{54}\left(110 - x^{(3)} - y^{(3)}\right) = \frac{1}{54}[110 - (2.492) - (3.685)] = 1.923$$

5^{th} Iteration :

$$x^{(5)} = \frac{1}{27}\left(85 - 6y^{(4)} + z^{(4)}\right) = \frac{1}{27}[85 - 6(3.545) + (1.923)] = 2.432$$

$$y^{(5)} = \frac{1}{15}\left(72 - 6x^{(4)} - 2z^{(4)}\right) = \frac{1}{15}[72 - 6(2.401) - 2(1.923)] = 3.583$$

$$z^{(5)} = \frac{1}{54}\left(110 - x^{(4)} - y^{(4)}\right) = \frac{1}{54}[110 - (2.401) - (3.545)] = 1.927$$

6^{th} Iteration :

$$x^{(6)} = \frac{1}{27}\left(85 - 6y^{(5)} + z^{(5)}\right) = \frac{1}{27}[85 - 6(3.583) + (1.927)] = 2.423$$

$$y^{(6)} = \frac{1}{15}\left(72 - 6x^{(5)} - 2z^{(5)}\right) = \frac{1}{15}[72 - 6(2.432) - 2(1.1927)] = 3.570$$

$$z^{(6)} = \frac{1}{54}\left(110 - x^{(5)} - y^{(5)}\right) = \frac{1}{54}[110 - (2.432) - (3.583)] = 1.926$$

7^{th} Iteration :

$$x^{(7)} = \frac{1}{27}\left(85 - 6y^{(6)} + z^{(6)}\right) = \frac{1}{27}[85 - 6(3.570) + (1.926)] = 2.426$$

$$y^{(7)} = \frac{1}{15}\left(72 - 6x^{(6)} - 2z^{(6)}\right) = \frac{1}{15}[72 - 6(2.423) - 2(1.926)] = 3.574$$

$$z^{(7)} = \frac{1}{54}\left(110 - x^{(6)} - y^{(6)}\right) = \frac{1}{54}[110 - (2.423) - (3.570)] = 1.926$$

8^{th} Iteration :

$$x^{(8)} = \frac{1}{27}\left(85 - 6y^{(7)} + z^{(7)}\right) = \frac{1}{27}[85 - 6(3.574) + (1.926)] = 2.425$$

$$y^{(8)} = \frac{1}{15}\left(72 - 6x^{(7)} - 2z^{(7)}\right) = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(8)} = \frac{1}{54}\left(110 - x^{(7)} - y^{(7)}\right) = \frac{1}{54}[110 - (2.426) - (6.547)] = 1.926$$

9^{nd} Iteration :

$$x^{(9)} = \frac{1}{27}\left(85 - 6y^{(8)} + z^{(8)}\right) = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(9)} = \frac{1}{15}\left(72 - 6x^{(8)} - 2z^{(8)}\right) = \frac{1}{15}[72 - 6(2.425) - 2(1.926)] = 3.573$$

$$z^{(9)} = \frac{1}{54}\left(110 - x^{(8)} - y^{(8)}\right) = \frac{1}{54}[110 - (2.425) - (3.573)] = 1.926$$

10^{nd} Iteration :

$$x^{(10)} = \frac{1}{27}\left(85 - 6y^{(9)} + z^{(9)}\right) = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(10)} = \frac{1}{15}\left(72 - 6x^{(9)} - 2z^{(9)}\right) = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(10)} = \frac{1}{54}\left(110 - x^{(9)} - y^{(9)}\right) = \frac{1}{54}[110 - (2.426) - (3.573)] = 1.926$$

Hence $\boldsymbol{x = 2.426}$, $\boldsymbol{y = 3.573}$ ***and*** $\boldsymbol{z = 1.926}$, correct to three decimal places.

Gauss Siedal Method :

Let the initial values be $\boldsymbol{y = 0}$, $\boldsymbol{z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{27}\left(85 - 6y^{(0)} + z^{(0)}\right) = \frac{1}{27}[85 - 6(0) + (0)] = 3.148$$

$$y^{(1)} = \frac{1}{15}\left(72 - 6x^{(1)} - 2z^{(0)}\right) = \frac{1}{15}[72 - 6(3.148) - 2(0)] = 3.541$$

$$z^{(1)} = \frac{1}{54}\left(110 - x^{(1)} - y^{(1)}\right) = \frac{1}{54}[110 - 3.148 - 3.541] = 1.913$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{27}\left(85 - 6y^{(1)} + z^{(1)}\right) = \frac{1}{27}[85 - 6(3.541) + (1.913)] = 2.432$$

$$y^{(2)} = \frac{1}{15}\left(72 - 6x^{(2)} - 2z^{(1)}\right) = \frac{1}{15}[72 - 6(2.432) - 2(1.913)] = 3.572$$

$$z^{(2)} = \frac{1}{54}\left(110 - x^{(2)} - y^{(2)}\right) = \frac{1}{54}[110 - (2.432) - (3.572)] = 1.926$$

3^{rd} Iteration :

$$x^{(3)} = \frac{1}{27}\left(85 - 6y^{(2)} + z^{(2)}\right) = \frac{1}{27}[85 - 6(3.572) + (1.926)] = 2.426$$

$$y^{(3)} = \frac{1}{15}\left(72 - 6x^{(3)} - 2z^{(2)}\right) = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(3)} = \frac{1}{54}\left(110 - x^{(3)} - y^{(3)}\right) = \frac{1}{54}[110 - (2.426) - (3.573)] = 1.926$$

4^{th} Iteration :

$$x^{(4)} = \frac{1}{27}\left(85 - 6y^{(3)} + z^{(3)}\right) = \frac{1}{27}[85 - 6(3.573) + (1.926)] = 2.426$$

$$y^{(4)} = \frac{1}{15}\left(72 - 6x^{(4)} - 2z^{(3)}\right) = \frac{1}{15}[72 - 6(2.426) - 2(1.926)] = 3.573$$

$$z^{(4)} = \frac{1}{54}\left(110 - x^{(4)} - y^{(4)}\right) = \frac{1}{54}[110 - (2.426) - (3.573)] = 1.926$$

Hence $\boldsymbol{x = 2.426,\ \ y = 3.573\ \ and\ \ z = 1.926}$, correct to three decimal places.

2. **Solve the following system of equations by Gauss-Jacobi and Gauss-Siedal method of Iteration.**

   $\boldsymbol{4x + 2y + z = 14,\ \ x + 5y - z = 10,\ \ \ x + y + 8z = 20.}$

   Solution :

$$4x + 2y + z = 14$$
$$x + 5y - z = 10$$
$$x + y + 8z = 20$$

Since,

$$|4| > |2| + |1|, \qquad |5| > |1| + |1|, \qquad |8| > |1| + |1|$$

So that we use Gauss iterative method,

Since the diagonal elements are dominant in the coefficient matrix, we rewrite x, y, z as follows

$$x = \frac{1}{4}(14 - 2y - z)$$

$$y = \frac{1}{5}(10 - x + z)$$

$$z = \frac{1}{8}(20 - x - y)$$

Gauss Jacobi Method :

Let the initial values be $\boldsymbol{x = 0, y = 0, z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{4}[14 - 2(0) - (0)] = 3.5$$

$$y^{(1)} = \frac{1}{5}[10-(0)+(0)] = 2$$

$$z^{(1)} = \frac{1}{8}[20-(0)-(0)] = 2.5$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{4}\left(14-2y^{(1)}-z^{(1)}\right) = \frac{1}{4}[14-2(2)-(2.5)] = 1.875$$

$$y^{(2)} = \frac{1}{5}\left(10-x^{(1)}+z^{(1)}\right) = \frac{1}{5}[10-(3.5)+(2.5)] = 1.8$$

$$z^{(2)} = \frac{1}{8}\left(20-x^{(1)}-y^{(1)}\right) = \frac{1}{8}[20-(3.5)-(2)] = 1.8125$$

We form the Iterations in the table

Iteration	x	y	z
1	3.5	2	2.5
2	1.875	1.8	1.8125
3	2.1093	1.9875	2.0406
4	1.9961	1.98626	1.9879
5	2.0098	1.9983	2.0022
6	2.0003	1.9984	1.9989
7	2.0010	1.99972	2.0001
8	2.0001	1.99982	1.9999
9	2.0001	1.99996	2.00000
10	2.0000	2.0000	2.0000

Hence the solution is $\boldsymbol{x = 2, \ y = 2 \ \ and \ \ z = 2.}$

Gauss Siedal Method :

Let the initial values be $\boldsymbol{y = 0, z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{4}\left(14-2y^{(0)}-z^{(0)}\right) = \frac{1}{4}[14-2(0)-(0)] = 3.5$$

$$y^{(1)} = \frac{1}{5}\left(10-x^{(1)}+z^{(0)}\right) = \frac{1}{5}[10-(3.5)+(0)] = 1.3$$

$$z^{(1)} = \frac{1}{8}\left(20-x^{(1)}-y^{(1)}\right) = \frac{1}{8}[20-(3.5)-(1.3)] = 1.9$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{4}\left(14-2y^{(1)}-z^{(1)}\right) = \frac{1}{4}[14-2(1.3)-(1.9)] = 2.375$$

$$y^{(2)} = \frac{1}{5}\left(10-x^{(2)}+z^{(1)}\right) = \frac{1}{5}[10-(2.375)+(1.9)] = 1.905$$

$$z^{(2)} = \frac{1}{8}\left(20-x^{(2)}-y^{(2)}\right) = \frac{1}{8}[20-(2.375)-(1.905)] = 1.965$$

We form the Iterations in the table

Iteration	x	y	z

1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.056	1.982	1.995
4	2.010	1.997	1.999
5	2.002	1.999	2
6	2.001	2	2
7	2	2	2
8	2	2	2

Hence the solution is $\boldsymbol{x = 2,\ \ y = 2\ \ and\ \ z = 2.}$

3. Solve the following system of equations by Gauss-Jacobi and Gauss-Siedal method of Iteration.

$\boldsymbol{10x - 5y - 2z = 3,\ \ 4x - 10y + 3z = -3,\ \ \ x + 6y + 10z = -3.}$

Solution :

$$10x - 5y - 2z = 3$$

$$4x - 10y + 3z = -3$$

$$x + 6y + 10z = -3$$

$Since,\ \ |10| > |5| + |2|\,,\ \ \ |10| > |4| + |3|\,,\ \ \ |10| > |1| + |6|$

So that we use Gauss iterative method,

Since the diagonal elements are dominant in the coefficient matrix, we rewrite x, y, z as follows

$$x = \frac{1}{10}(3 + 5y + 2z)$$

$$y = \frac{1}{10}(3 + 4x + 3z)$$

$$z = \frac{1}{10}(-3 - x - 6y)$$

Gauss Jacobi Method :

Let the initial values be $\boldsymbol{x = 0, y = 0, z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2\,(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3\,(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] = -0.3$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{10}\left(3 + 5y^{(1)} + 2z^{(1)}\right) = \frac{1}{10}[3 + 5(0.3) + 2\,(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}\left(3 + 4x^{(1)} + 3z^{(1)}\right) = \frac{1}{10}[3 + 4(0.3) + 3\,(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}\left(-3 - x^{(1)} - 6y^{(1)}\right) = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = -0.51$$

$\boldsymbol{Iteration}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
1	0.3	0.3	-0.3

2	0.39	0.33	− 0.51
3	0.363	0.303	− 0.537
4	0.3441	0.2841	− 0.5181
5	0.33843	0.2822	− 0.50487
6	0.340126	0.283911	− 0.503163
7	0.3413229	0.2851015	− 0.2043592
8	0.34167891	0.2852214	− 0.50519319
9	0.341572062	0.285113607	− 0.505300731

Hence $\boldsymbol{x = 0.342,\ \ y = 0.285\ \ and\ \ z = -0.505}$ correct to three decimal places.

Gauss Siedal Method :

Let the initial values be $\boldsymbol{y = 0,\ z = 0}$

1^{st} Iteration :

$$x^{(1)} = \frac{1}{4}\left(3 + 5y^{(0)} + 2z^{(0)}\right) = \frac{1}{10}[3 + 5(0) + 2\,(0)] = 0.3$$

$$y^{(1)} = \frac{1}{5}\left(3 + 4x^{(1)} + 3z^{(0)}\right) = \frac{1}{10}[3 + 4(0.3) + 3\,(0)] = 0.42$$

$$z^{(1)} = \frac{1}{8}\left(-3 - x^{(1)} - y^{(1)}\right) = \frac{1}{10}[-3 - (0.3) - 6(0.42)] = -0.582$$

2^{nd} Iteration :

$$x^{(2)} = \frac{1}{4}\left(3 + 5y^{(1)} + 2z^{(1)}\right) = \frac{1}{10}[3 + 5(0.42) + 2\,(-0.582)] = 0.3936$$

$$y^{(2)} = \frac{1}{5}\left(3 + 4x^{(2)} + 3z^{(1)}\right) = \frac{1}{10}[3 + 4(0.3936) + 3\,(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{8}\left(-3 - x^{(2)} - 6y^{(2)}\right) = \frac{1}{10}[-3 - (0.39396) - 6(0.28284)] = -0.509064$$

Iteration	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
1	0.3	0.42	− 0.582
2	0.3936	0.28284	− 0.509064
3	0.3396072	0.28312364	− 0.503834928
4	0.34079485	0.28516746	− 0.50517996
5	0.3415547	0.28506792	− 0.505196229
6	0.341497	0.2850390	− 0.5051728
7	0.341489	0.28504212	− 0.5051737

Hence $\boldsymbol{x = 0.342,\ \ y = 0.285\ \ and\ \ z = -0.505}$ correct to three decimal places.

4. Solve the following system of equations by Gauss-Jacobi and Gauss-Siedal method of Iteration.

   $8\,x - 3y + 2z = 20,\ \ 4x + 11\,y - z = 33,\ \ \ 6\,x + 3\,y + 12\,z = 35$

   Solution :

Let the given system be

$$8\,x - 3y + 2z = 20$$

$$4x + 11\,y - z = 33$$

$$6\,x + 3\,y + 12\,z = 35$$

Since,

$$|8| > |3| + |2|, \quad |11| > |4| + |1|, \quad |12| > |6| + |3|$$

So that we use Gauss iterative method,

Since the diagonal elements are dominant in the coefficient matrix, we rewrite x, y, z as follows

$$x = \frac{1}{8}(20 + 3\,y - 2\,z)$$

$$y = \frac{1}{11}(33 - 4\,x + z)$$

$$z = \frac{1}{12}(35 - 6\,x - 3\,y)$$

Gauss Jacobi Method :

Let the initial values be $\boldsymbol{x = 0, y = 0, z = 0}$

$\boldsymbol{Iteration}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
1	2.5	3.0	2.916666
2	2.895833	2.356060	0.916666
3	3.154356	2.030303	0.879735
4	3.041430	1.930937	0.831913
5	3.016873	1.969654	0.912717
6	3.010441	1.985930	0.915817
7	3.015770	1.988550	0.914964
8	3.016946	1.986535	0.911644
9	3.017039	1.985805	0.911560
10	3.016786	1.985764	0.911696

Gauss Siedal Method :

Let the initial values be $\boldsymbol{x = 0, y = 0}$

$\boldsymbol{Iteration}$	$\boldsymbol{x}$	$\boldsymbol{y}$	$\boldsymbol{z}$
1	2.5	2.090909	1.143939
2	2.998106	2.013774	0.914170
3	3.026623	1.982516	0.907726
4	3.016512	1.985607	0.912009
5	3.01660	1.985964	0.911876
6	3.016767	1.985892	0.911810
7	3.016757	1.985889	0.911816

5. **Solve by Gauss – Siedal method correct to four decimal places.**

   $\boldsymbol{x - 2y = -3 \;\; and \;\; 2x + 25y = 15}$.

   Solution :

$$x - 2y = -3 \;\; and \;\; 2x + 25y = 15$$

$$x = 2y + 3$$

$$y = \frac{1}{25}[15 - 2x]$$

Let the initial value be $\boldsymbol{y = 0}$

1^{st} Iteration :

$$x^{(1)} = -3 + 2y = -3 + 2[0] = -3$$

$$y^{(1)} = \frac{1}{25}\left(15 - 2x^{(1)}\right) = \frac{1}{25}[15 - 2(-3)] = 0.84$$

2^{nd} Iteration :

$$x^{(2)} = -3 + 2y^{(1)} = -3 + 2[0.84] = -1.32$$

$$y^{(2)} = \frac{1}{25}\left(15 - 2x^{(2)}\right) = \frac{1}{25}[15 - 2(-1.32)] = 0.7056$$

We form the table as follows

$Iteration$	x	y
1	-3	0.84
2	-1.32	0.7056
3	-1.5888	0.7271
4	-1.5858	0.7237
5	-1.5526	0.7242
6	-1.5516	0.7241
7	-1.5518	0.7241
8	-1.5518	0.7241

Hence $\boldsymbol{x = -1.5518,\ y = 0.7241}$ correct to four decimal places.

EIGEN VALES OF A MATRIX BY POWER METHOD

1. **Using power method find the all Eigen value and the corresponding Eigen vector of the matrix** $\boldsymbol{A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$.

Solution : Let $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\,X_2$$

$$A\,X_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7\,X_3$$

$$A\,X_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.574 \\ 1.8572 \\ 0 \end{bmatrix} = 3.574\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = 3.574\,X_4$$

$$A\,X_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = 4.12\,X_5$$

$$A\,X_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = 3.9706\,X_6$$

$$A\,X_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 4.0072\,X_7$$

$$A\,X_7 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.9982 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 3.9982\,X_8$$

$$A\,X_8 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.50 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5000 \\ 0 \end{bmatrix} = 4\,X_9$$

∴ The dominant **Eigen value** = 4.

Corresponding **Eigen vector** is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$.

To find the Second Eigen value :

Let $B = A - \lambda I \;\; \Rightarrow \;\; B = A - 4\,I.$

$$B = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

We need to find the dominant Eigen value for the matrix B.

Let $Y_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$B\,Y_1 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -\,3\,Y_2$$

$$B\,Y_2 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -\,5\,Y_3$$

$$B\,Y_3 = \begin{bmatrix} -3 & 6 & 1 \\ 1 & -2 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = \begin{bmatrix} -5 \\ 1.6666 \\ 0 \end{bmatrix} = -5 \begin{bmatrix} 1 \\ -0.3333 \\ 0 \end{bmatrix} = -\,5\,Y_4$$

∴ The dominant Eigen value for B = - 5.

Sum of Eigen values = Trace of the matrix A

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3$

$\lambda_1 + 4 - 5 = 6 \;\; \Longrightarrow \;\; \lambda_1 = 7$

∴ The three Eigen values are - 5, 4 & 7 .

The Eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$.

2. Using power method find the all Eigen value and the corresponding Eigen vector of the matrix $\boldsymbol{A} = \begin{bmatrix} \mathbf{5} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{5} \end{bmatrix}$.

Solution : Let $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = 1\,X_2$$

$$A\,X_2 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = 5.2\,X_3$$

$$A\,X_3 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.3846 \end{bmatrix} = \begin{bmatrix} 5.3846 \\ 0 \\ 2.9231 \end{bmatrix} = 5.3846\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = 5.3846\,X_4$$

$$A\,X_4 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.5429 \end{bmatrix} = \begin{bmatrix} 5.5429 \\ 0 \\ 3.7143 \end{bmatrix} = 5.5429\begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = 5.5429\,X_5$$

$$A\,X_5 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.6701 \end{bmatrix} = \begin{bmatrix} 5.6701 \\ 0 \\ 4.3505 \end{bmatrix} = 5.6701\begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = 5.6701\,X_6$$

$$A\,X_6 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.7672 \end{bmatrix} = \begin{bmatrix} 5.7672 \\ 0 \\ 4.8360 \end{bmatrix} = 5.7672\begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = 5.7672\,X_7$$

$$A\,X_7 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8385 \end{bmatrix} = \begin{bmatrix} 5.8385 \\ 0 \\ 5.1927 \end{bmatrix} = 5.8385\begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = 5.8385\,X_8$$

$$A\,X_8 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.8894 \end{bmatrix} = \begin{bmatrix} 5.8894 \\ 0 \\ 5.4470 \end{bmatrix} = 5.8894\begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = 5.8894\,X_9$$

$$A\,X_9 = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.9249 \end{bmatrix} = \begin{bmatrix} 5.9249 \\ 0 \\ 5.6244 \end{bmatrix} = 5.9249\begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = 5.9249\,X_{10}$$

$$A\,X_{10} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9493 \end{bmatrix} = \begin{bmatrix} 5.9493 \\ 0 \\ 5.7465 \end{bmatrix} = 5.9493 \begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = 5.9493\,X_{11}$$

$$A\,X_{11} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9659 \end{bmatrix} = \begin{bmatrix} 5.9659 \\ 0 \\ 5.8296 \end{bmatrix} = 5.9659 \begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = 5.9659\,X_{12}$$

$$A\,X_{12} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9771 \end{bmatrix} = \begin{bmatrix} 5.9771 \\ 0 \\ 5.8857 \end{bmatrix} = 5.9771 \begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = 5.9771\,X_{13}$$

$$A\,X_{13} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9847 \end{bmatrix} = \begin{bmatrix} 5.9847 \\ 0 \\ 5.9236 \end{bmatrix} = 5.9847 \begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = 5.9847\,X_{14}$$

$$A\,X_{14} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9898 \end{bmatrix} = \begin{bmatrix} 5.9898 \\ 0 \\ 5.9489 \end{bmatrix} = 5.9898 \begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = 5.9898\,X_{15}$$

$$A\,X_{15} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9932 \end{bmatrix} = \begin{bmatrix} 5.9932 \\ 0 \\ 5.9659 \end{bmatrix} = 5.9932 \begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = 5.9932\,X_{16}$$

$$A\,X_{16} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9954 \end{bmatrix} = \begin{bmatrix} 5.9954 \\ 0 \\ 5.9772 \end{bmatrix} = 5.9954 \begin{bmatrix} 1 \\ 0 \\ 0.9970 \end{bmatrix} = 5.9954\,X_{17}$$

$$A\,X_{17} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.9970 \end{bmatrix} = \begin{bmatrix} 5.9970 \\ 0 \\ 5.9848 \end{bmatrix} = 5.9974 \begin{bmatrix} 1 \\ 0 \\ 0.9980 \end{bmatrix}$$

∴ The dominant **Eigen value = 6 (app).**

Corresponding **Eigen vector is** $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $(\boldsymbol{app})$.

To find the Second Eigen value:

Let $B = A - \lambda\, I \;\; \Longrightarrow \;\; B = A - 4\, I.$

$$B = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} - 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

We need to find the dominant Eigen value for the matrix B.

Let $Y_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

$$B\,Y_1 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -\,1\,Y_2$$

$$B\,Y_2 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -\,2\,Y_3$$

$$B\,Y_3 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -8 & 0 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

∴ The dominant Eigen value for B = - 2.

Sum of Eigen values = Trace of the matrix A

$\lambda_1 + \lambda_2 + \lambda_3 = 5 - 2 + 5$

$\lambda_1 + 6 - 2 = 8 \;\; \Longrightarrow \;\; \lambda_1 = 4$

∴ The three Eigen values are - 2, 4 & 6 .

The Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

3. **Find the largest Eigen value and the corresponding Eigen vector of the matrix $\begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$.**

Solution : Let $X_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = 25\,X_2$$

$$A\,X_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = 25.2\,X_3$$

$$A\,X_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = 25.18\,X_4$$

$$A\,X_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.12 \\ 1.72 \end{bmatrix} = 25.18\begin{bmatrix} 1.00 \\ 0.04 \\ 0.07 \end{bmatrix}$$

∴ The dominant Eigen value = 25.18 (app).

Corresponding Eigen vector is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ (app).

4. **Using power method find the dominant Eigen value and the corresponding Eigen vector of the matrix**

$\boldsymbol{A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}}$.

Solution : Let $X_0 = \begin{bmatrix}1\\0\end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\1\end{bmatrix} = 4\begin{bmatrix}1\\0.25\end{bmatrix} = 4\,X_2$$

$$A\,X_2 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.25\end{bmatrix} = \begin{bmatrix}4.25\\1.75\end{bmatrix} = 4.25\begin{bmatrix}1\\0.4118\end{bmatrix} = 4.25\,X_3$$

$$A\,X_3 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.4118\end{bmatrix} = \begin{bmatrix}4.4118\\2.2352\end{bmatrix} = 4.4118\begin{bmatrix}1\\0.5066\end{bmatrix} = 4.4118\,X_4$$

$$A\,X_4 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.5066\end{bmatrix} = \begin{bmatrix}4.5066\\2.5199\end{bmatrix} = 4.5066\begin{bmatrix}1\\0.5591\end{bmatrix} = 4.5066\,X_5$$

$$A\,X_5 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.5591\end{bmatrix} = \begin{bmatrix}4.5591\\2.677\end{bmatrix} = 4.5591\begin{bmatrix}1\\0.5871\end{bmatrix} = 4.5591\,X_6$$

$$A\,X_6 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.5871\end{bmatrix} = \begin{bmatrix}4.5871\\2.7613\end{bmatrix} = 4.5871\begin{bmatrix}1\\0.6019\end{bmatrix} = 4.5871\,X_7$$

$$A\,X_7 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.6019\end{bmatrix} = \begin{bmatrix}4.6019\\2.8057\end{bmatrix} = 4.6019\begin{bmatrix}1\\0.6096\end{bmatrix} = 4.6019\,X_8$$

$$A\,X_8 = \begin{bmatrix}4 & 1\\1 & 3\end{bmatrix}\begin{bmatrix}1\\0.6096\end{bmatrix} = \begin{bmatrix}4.6096\\2.8288\end{bmatrix} = 4.6096\begin{bmatrix}1\\0.6137\end{bmatrix} = 4.6096\,X_9$$

∴ The dominant Eigen value = 4.60

Corresponding Eigen vector is $\begin{bmatrix}1\\0.6137\end{bmatrix}$.

5. Find numerically largest e Eigen value and the corresponding Eigen vector of the matrix by power method $\begin{bmatrix}1 & -3 & 2\\4 & 4 & -1\\6 & 3 & 5\end{bmatrix}$.

Solution : Let $X_0 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ be the initial vector.

Therefore,

$$A\,X_1 = \begin{bmatrix}1 & -3 & 2\\4 & 4 & -1\\6 & 3 & 5\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\4\\6\end{bmatrix} = 6\begin{bmatrix}0.167\\0.667\\1\end{bmatrix} = 6\,X_2$$

$$A\,X_2 = \begin{bmatrix}1 & -3 & 2\\4 & 4 & -1\\6 & 3 & 5\end{bmatrix}\begin{bmatrix}0.167\\0.667\\1\end{bmatrix} = \begin{bmatrix}0.166\\2.336\\8.003\end{bmatrix} = 8.003\begin{bmatrix}0.021\\0.292\\1\end{bmatrix} = 8.003\,X_3$$

$$A\,X_3 = \begin{bmatrix}1.145\\0.252\\6.002\end{bmatrix} = 6.002\begin{bmatrix}0.191\\0.042\\1\end{bmatrix} = 6.002\,X_4$$

$$A\,X_4 = \begin{bmatrix}2.065\\-0.068\\6.272\end{bmatrix} = 6.272\begin{bmatrix}0.329\\-0.011\\1\end{bmatrix} = 6.272\,X_5$$

$$A\,X_5 = \begin{bmatrix} 2.362 \\ 0.272 \\ 6.941 \end{bmatrix} = 6.941 \begin{bmatrix} 0.34 \\ 0.039 \\ 1 \end{bmatrix} = 6.941\,X_6$$

$$A\,X_6 = \begin{bmatrix} 2.223 \\ 0.516 \\ 7.157 \end{bmatrix} = 7.157 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = 7.157\;X_7$$

$$A\,X_7 = \begin{bmatrix} 2.065 \\ 0.532 \\ 7.082 \end{bmatrix} = 7.082 \begin{bmatrix} 0.296 \\ 0.075 \\ 1 \end{bmatrix} = 7.082\,X_8$$

$$A\,X_8 = \begin{bmatrix} 2.071 \\ 0.484 \\ 7.001 \end{bmatrix} = 7.001 \begin{bmatrix} 0.296 \\ 0.069 \\ 1 \end{bmatrix} = 7.001\,X_9$$

$$A\,X_9 = \begin{bmatrix} 2.089 \\ 0.46 \\ 6.983 \end{bmatrix} = 6.983 \begin{bmatrix} 0.296 \\ 0.066 \\ 1 \end{bmatrix} = 6.983\,X_{10}$$

$$A\,X_{10} = \begin{bmatrix} 2.101 \\ 0.46 \\ 6.992 \end{bmatrix} = 6.992 \begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix} = 6.992\,X_{11}$$

$$A\,X_{11} = \begin{bmatrix} 2.102 \\ 0.464 \\ 6.998 \end{bmatrix} = 6.998 \begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix} = 6.998\,X_{12}$$

$$A\,X_{12} = \begin{bmatrix} 2.102 \\ 0.464 \\ 6.998 \end{bmatrix} = 6.998 \begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix} = 6.998\,X_{13}$$

∴ The Eigen value = 6.998.

Corresponding Eigen vector is $\begin{bmatrix} 0.3 \\ 0.066 \\ 1 \end{bmatrix}$.

INVERSE OF A MATRIX BY GAUSS JORDAN METHOD

Example : 1

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solution:

We know that $[A, I] = [I, A^{-1}]$

Now, $[A, I] = \begin{bmatrix} 1 & 3 & 3 & \vdots & 1 & 0 & 0 \\ 1 & 4 & 3 & \vdots & 0 & 1 & 0 \\ 1 & 3 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}$

Now, we need to make $[A.I]$ as a diagonal matrix.

$Fix\ the\ first\ row, change\ second\ and\ third\ row\ by\ using\ first\ row.$

$$[A,I] \sim \begin{bmatrix} 1 & 3 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow R_2 - R_1 \\ R_3 \Leftrightarrow R_3 - R_1 \end{matrix}$$

Fix the third row, change first and second row by using third row.

$$[A,I] \sim \begin{bmatrix} 1 & 3 & 0 & \vdots & 4 & 0 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix} \quad R_1 \Leftrightarrow R_1 - 3R_3$$

Fix the second & third row, change first by using second row.

$$[A,I] \sim \begin{bmatrix} 1 & 0 & 0 & \vdots & 7 & -3 & -3 \\ 0 & 1 & 0 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & 1 \end{bmatrix} = [I, A^{-1}] \quad R_1 \Leftrightarrow R_1 - 3R_2$$

$$A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Verification :

W.k.t $A\,A^{-1} = I$ $\quad \Rightarrow \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} * \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example : 1

Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$

Solution:

We know that $[A,I] = [I, A^{-1}]$

Now, $[A,I] = \begin{bmatrix} 2 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 1 & -1 & 1 & \vdots & 0 & 1 & 0 \\ 4 & 2 & -3 & \vdots & 0 & 0 & 1 \end{bmatrix}$

Now, we need to make $[A,I]$ as a diagonal matrix.

Fix the first row, change second and third row by using first row.

$$[A,I] \sim \begin{bmatrix} 2 & 1 & 1 & \vdots & 1 & 0 & 0 \\ 0 & -3 & 1 & \vdots & -1 & 2 & 0 \\ 0 & 0 & -10 & \vdots & -4 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \Leftrightarrow 2\,R_2 - R_1 \\ R_3 \Leftrightarrow 2\,R_3 - 4\,R_1 \end{matrix}$$

Fix the third row, change first and second row by using third row.

$$[A,I] \sim \begin{bmatrix} -20 & -10 & 0 & \vdots & -6 & 0 & -2 \\ 0 & 30 & 0 & \vdots & 14 & -20 & -2 \\ 0 & 0 & -10 & \vdots & -4 & 0 & 2 \end{bmatrix} \quad \begin{matrix} R_1 \Leftrightarrow -10\,R_1 - R_3 \\ R_2 \Leftrightarrow -10\,R_2 - R_3 \end{matrix}$$

Fix the second & third row, change first by using second row.

$$[A,I] \sim \begin{bmatrix} -600 & 0 & 0 & \vdots & -40 & -200 & -80 \\ 0 & 30 & 0 & \vdots & 14 & -20 & -2 \\ 0 & 0 & -10 & \vdots & -4 & 0 & 2 \end{bmatrix} \quad R_1 \Leftrightarrow 30\,R_1 - (-10)R_2$$

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-40}{-600} & \frac{-200}{-600} & \frac{-80}{-600} \\ 0 & 1 & 0 & \frac{14}{30} & \frac{-20}{30} & \frac{-2}{30} \\ 0 & 0 & 1 & \frac{-4}{-10} & 0 & \frac{2}{-10} \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow R_1/-600 \\ R_2 \Leftrightarrow R_2/30 \\ R_3 \Leftrightarrow R_3/-10 \end{array}$$

$$A^{-1} = \begin{bmatrix} 1/15 & 1/3 & 2/15 \\ 7/15 & -2/3 & -1/15 \\ 2/5 & 0 & -1/5 \end{bmatrix}$$

Verification :

W.k.t $A\,A^{-1} = I$ $\quad \Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix} * \begin{bmatrix} 1/15 & 1/3 & 2/15 \\ 7/15 & -2/3 & -1/15 \\ 2/5 & 0 & -1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example : 3

Find the inverse of the matrix $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$

Solution:

We know that $[A,I] = [I, A^{-1}]$

Now, $[A,I] = \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$

Now, we need to make $[A.I]$ as a diagonal matrix.

Fix the first row, change second and third row by using first row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & 10 & -8 & -2 & 4 & 0 \\ 0 & -9 & 6 & -1 & 0 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \Leftrightarrow 4\,R_2 - 2\,R_1 \\ R_3 \Leftrightarrow 4\,R_3 - 1\,R_1 \end{array}$$

Fix the first row & second row, change third row by using second row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & 10 & -8 & -2 & 4 & 0 \\ 0 & 0 & -12 & -28 & 36 & 40 \end{array}\right] \quad R_3 \Leftrightarrow 10\,R_3 - (-9)\,R_2$$

Fix the third row, change first and second row by using third row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} -48 & -12 & 0 & 44 & -72 & -80 \\ 0 & -120 & 0 & -200 & 240 & 320 \\ 0 & 0 & -12 & -28 & 36 & 40 \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow -12\,R_1 - 2\,R_3 \\ R_2 \Leftrightarrow -12\,R_2 - (-8)R_3 \end{array}$$

Fix the second & third row, change first by using second row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 5760 & 0 & 0 & -7680 & 11520 & 13440 \\ 0 & -120 & 0 & -200 & 240 & 320 \\ 0 & 0 & -12 & -28 & 36 & 40 \end{array}\right] \quad R_1 \Leftrightarrow -120\,R_1 - (-12)R_2$$

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-7680}{5760} & \frac{11520}{5760} & \frac{13440}{5760} \\ 0 & 1 & 0 & \frac{-200}{-120} & \frac{240}{-120} & \frac{320}{-120} \\ 0 & 0 & 1 & \frac{-28}{-12} & \frac{36}{-12} & \frac{40}{-12} \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow R_1/960 \\ R_2 \Leftrightarrow R_2/-120 \\ R_3 \Leftrightarrow R_3/-12 \end{array}$$

$$A^{-1} = \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -2 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix}$$

Verification :

W.k.t $A\,A^{-1} = I$ $\quad \Rightarrow \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix} * \begin{bmatrix} -4/3 & 2 & 7/3 \\ 5/3 & -2 & -8/3 \\ 7/3 & -3 & -10/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example : 4

Find the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 2 & 5 \\ 1 & -1 & 0 \end{bmatrix}$

Solution:

When we finding the inverse of a matrix A, the diagonal elements should not be zero. If its zero, then re-arrange the given matrix A. That is

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} \; (Correct\ form)$$

We know that $[A,I] = [I,A^{-1}]$

Now, $[A,I] = \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$

Now, we need to make $[A.I]$ as a diagonal matrix.

Fix the first row, change second and third row by using first row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 2 & 0 \\ 0 & 4 & 7 & -3 & 0 & 2 \end{array}\right] \quad \begin{array}{l} R_2 \Leftrightarrow 2\,R_2 - 1\,R_1 \\ R_3 \Leftrightarrow 2\,R_3 - 3\,R_1 \end{array}$$

Fix the first row & second row, change third row by using second row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 2 & 0 \\ 0 & 0 & -10 & 10 & -8 & -4 \end{array}\right] \quad R_3 \Leftrightarrow -2\,R_3 - 4\,R_2$$

Fix the third row, change first and second row by using third row.

$$[A,I] \sim \left[\begin{array}{ccc|ccc} -20 & 0 & 0 & -20 & 8 & 4 \\ 0 & 20 & 0 & 20 & -28 & -4 \\ 0 & 0 & -10 & 10 & -8 & -4 \end{array}\right] \quad \begin{array}{l} R_1 \Leftrightarrow -10\,R_1 - 1\,R_3 \\ R_2 \Leftrightarrow -10\,R_2 - (-1)R_3 \end{array}$$

$$[A,I] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -20/-20 & 8/-20 & 4/-20 \\ 0 & 1 & 0 & 20/20 & -28/20 & -4/20 \\ 0 & 0 & 1 & 10/-10 & -8/-10 & -4/-10 \end{array}\right] \quad \begin{array}{c} R_1 \Leftrightarrow R_1/-20 \\ R_2 \Leftrightarrow R_2/20 \\ R_3 \Leftrightarrow R_3/-10 \end{array}$$

$$A^{-1} = \begin{bmatrix} 1 & -2/5 & -1/5 \\ 1 & 7/5 & -1/5 \\ -1 & 4/5 & 2/5 \end{bmatrix}$$

Verification :

W.k.t $A\,A^{-1} = I$ $\quad \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} * \begin{bmatrix} 1 & -2/5 & -1/5 \\ 1 & 7/5 & -1/5 \\ -1 & 4/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

METHOD OF FALSE POSITION OR REGULA FALSI METHOD

Example – 1 : Solve for a positive root of $\boldsymbol{x^3 - 2x - 5 = 0}$ by regula falsi method.

Solution:

Given $f(x) = x^3 - 2x - 5.$

Now, $f(0) = (0)^3 - 2(0) - 5 = -5 \quad (-ve)$

$f(1) = (1)^3 - 2(1) - 5 = -6 \quad (-ve)$

$f(2) = (2)^3 - 2(2) - 5 = -1 \quad (-ve)$

$f(3) = (3)^3 - 2(3) - 5 = 16 \quad (+ve)$

$\therefore \quad The\ approximate\ root\ lies\ b/w\ \mathbf{2}\ (-\boldsymbol{ve}) \quad \& \quad 3\,(+ve)\,.$

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{1}, \mathbf{2})$

Now $\quad x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

$$x_1 = \frac{2\,f(3) - 3\,f(2)}{f(3) - f(2)} = \frac{2\,[16] - 3\,[-1]}{[16] - [-1]} = 2.058824$$

$x_1 = 2.058824.$

Now $\quad f(x_1) = f(2.058824) = (2.058824)^3 - 2(2.058824) - 5 = -0.390 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.058824

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{2.058824},\ \mathbf{2})$

$$x_2 = \frac{2.058824\,f(2) - 3\,f(2.058824)}{f(2) - f(2.058824)} = 2.081264$$

Now $f(x_2) = f(2.081264) = (2.081264)^3 - 2(2.081264) - 5 = -0.14 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.081264

$\therefore \quad (\boldsymbol{a,b}) = (\mathbf{2.081264,\ \ 2})$

$$x_3 = \frac{2.081264\, f(2) - 3\, f(2.081264)}{f(2) - f(2.081264)} = 2.089639$$

Now $f(x_3) = f(2.089639) = -0.054 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.089639

$\therefore \quad (\boldsymbol{a,b}) = (\mathbf{2.089639,\ \ 2})$

$$x_4 = \frac{2.089639 f(2) - 3\, f(2.089639)}{f(2) - f(2.089639)} = 2.092740$$

Now $f(x_4) = f(2.09274) = -0.020 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.09274

$\therefore \quad (\boldsymbol{a,b}) = (\mathbf{2.09274,\ \ 2})$

$$x_5 = \frac{2.09274\, f(2) - 3\, f(2.09274)}{f(2) - f(2.09274)} = 2.093884$$

Now $f(x_5) = f(2.093884) = -0.007 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.093884

$\therefore \quad (\boldsymbol{a,b}) = (\mathbf{2.093884,\ \ 2})$

$$x_6 = \frac{2.093884\, f(2) - 3\, f(2.093884)}{f(2) - f(2.093884)} = 2.094306$$

Now $f(x_6) = f(2.094306) = -0.007 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.094306

$\therefore \quad (\boldsymbol{a,b}) = (\mathbf{2.094306,\ \ 2})$

$$x_7 = \frac{2.094306\, f(2) - 3\, f(2.094306)}{f(2) - f(2.094306)} = 2.094461$$

$\therefore$ The Root of the given equation is $\mathbf{2.094}$ $(\boldsymbol{Correct\ to\ three\ decimal\ places})$.

Example – 2: Solve for a positive root of $\boldsymbol{xe^x = 2}$ by the method of false position.

Solution:

Given $f(x) = xe^x - 2$.

Now, $f(0) = (0)e^{0} - 2 = -2 \quad (-ve)$

$f(1) = (1)e^{1} - 2 = 0.718 \quad (+ve)$

$\therefore$ *The approximate root lies b/w* $\mathbf{0} \ (-\boldsymbol{ve})$ & $\mathbf{1}\ (+\boldsymbol{ve})$.

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0}, \mathbf{1})$

Now $\quad x_1 = \dfrac{a\,f(b) - b\,f(a)}{f(b) - f(a)}$

$$x_1 = \frac{0\,f(1) - 1\,f(0)}{f(1) - f(0)} = \frac{0[0.71828] - 3\,[-2]}{[0.71828] - [-2]} = 0.735759$$

$x_1 = 0.735759.$

Now $\quad f(x_1) = f(0.735759) = 0.735759\,e^{0.735759} - 2 = -0.46 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 0.735759

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0.735759},\ \mathbf{1})$

$$x_2 = \frac{0.735759\,f(1) - 1\,f(0.735759)}{f(1) - f(0.735759)} = 0.839521$$

Now $\quad f(x_2) = f(0.839521) = -0.05 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 0.839521

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0.839521},\ \mathbf{1})$

$$x_3 = \frac{0.839521\,f(1) - 1\,f(0.839521)}{f(1) - f(0.839521)} = 0.851184$$

Now $\quad f(x_3) = f(0.851184) = -0.0061 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 0.851184

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0.851184},\ \mathbf{1})$

$$x_4 = \frac{0.851184 f(1) - 1\,f(0.851184)}{f(1) - f(0.851184)} = 0.852452$$

Now $\quad f(x_4) = f(0.852452) = -0..020 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 0.852452

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0.852452},\ \mathbf{2})$

$$x_5 = \frac{0.852452\,f(1) - 1\,f(0.852452)}{f(1) - f(0.852452)} = 0.85261$$

Now $f(x_5) = f(0.85261) = -0.000019 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 0.85261

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{0.85261}, \ \mathbf{2})$

$$x_6 = \frac{0.85261\, f(1) - 1\, f(0.85261)}{f1 - f(0.85261)} = 0.85261$$

$\therefore$ The Root of the given equation is $\mathbf{0.85261}$ $(\boldsymbol{Correct\ to\ four\ decimal\ places}).$

Example- 3: Solve for a positive root of $\boldsymbol{x} \log_{10} \boldsymbol{x} - \mathbf{1.2} = \mathbf{0}$ by regula falsi method.

Solution:

Given $f(x) = x \log_{10} x - 1.2.$

Now, $f(0) = (0) \log_{10}(0) - 1.2 = -1.2 \quad (-ve)$

$f(1) = (1) \log_{10}(1) - 1.2 = -1.2 \quad (-ve)$

$f(2) = (2) \log_{10}(2) - 1.2 = -0.59 \quad (-ve)$

$f(3) = (3) \log_{10}(3) - 1.2 = 0.23 \quad (+ve)$

$\therefore \quad The\ approximate\ root\ lies\ b/w\ \mathbf{2}\ (\boldsymbol{-ve})\ \&\ \mathbf{3}\ (\boldsymbol{+ve}).$

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{1}, \mathbf{2})$

Now $\quad x_1 = \dfrac{a\, f(b) - b\, f(a)}{f(b) - f(a)}$

$$x_1 = \frac{2\, f(3) - 3\, f(2)}{f(3) - f(2)} = \frac{2\,[0.23136] - 3\,[-0.59794]}{[0.23136]{-}{-}0.59794} = 2.721014$$

$x_1 = 2.721014.$

Now $f(x_1) = f(2.721014) = (2.721014) \log_{10}(2.721014) - 1.2 = -0.01 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.721014

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{2.721014}, \ \mathbf{2})$

$$x_2 = \frac{2.721014\, f(2) - 3\, f(2.721014)}{f(2) - f(2.721014)} = 2.740211$$

Now $f(x_2) = f(2.7402) = (2.7402) \log_{10}(2.7402) - 1.2 = -0.00038 \quad (-ve)$

$\therefore$ we replace the $(-ve)$ value by 2.7402

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{2.7402}, \ \mathbf{2})$

$$x_3 = \frac{2.7402\ f(2) - 3\,f(2.7402)}{f(2) - f(2.7402)} = 2.740627$$

Now $f(x_3) = f(2.7406) = 0.00011 \quad (+ve)$

$\therefore$ we replace the $(+ve)$ value by 2.7406

$\therefore \quad (\boldsymbol{a}, \boldsymbol{b}) = (\mathbf{2.7402}, \ \mathbf{2.7406})$

$$x_4 = \frac{2.7402\,f(2.7406) - 2.7406\,f(2.7402)}{f(2.7406) - f(2.7402)} = 2.7405$$

$\therefore$ The Root of the given equation is $\mathbf{2.094}$ $(\boldsymbol{Correct\ to\ three\ decimal\ places}).$