1. Define statement function of one variable. When it will become a statement?
   Statement function is an expression containing symbols and an individual variable. It becomes a statement when the variable is replaced by particular value.

2. Use quantifiers to express the associative law for multiplication of real numbers.
   $\forall x\ \forall y\ \forall z\ ((x.y).z = x.(y.z))$ where the universe of discourse for x, y and z is the set of real numbers.

3. Let the universe of discourse be $E = \{5,6,7\}$. Let $A = \{5,6\}$ and $B = \{6,7\}$.
   $Let\ P(x): x\ is\ in\ A, Q(x): x\ is\ in\ B\ and\ R(x,y): x + y < 12.$
   $Find\ the\ truth\ values\ of\ (\exists x)(P(x) \rightarrow Q(x)) \rightarrow R(5,6)$
   Solution:
   $R(5,6)$ is true.
   $P(5)$ is true and $Q(5)$ is false
   $P(5) \rightarrow Q(5)$ is false
   $P(6)$ is true and $Q(6)$ is true.
   $P(6) \rightarrow Q(6)$ is true.
   $P(7)$ is false and $Q(7)$ is true.
   $P(7) \rightarrow Q(7)$ is false.
   $(\exists x)(P(x) \rightarrow Q(x)$ is true.
   Hence $(\exists x)(P(x) \rightarrow Q(x)) \rightarrow R(5,6)$ is true

4. Give an example in which $(\exists x)P(x) \rightarrow (\exists x)Q(x)$ is false.
   Let the universe of discourse be $E = \{3,4,5\}$
   Let $P(x): x < 5$; $Q(x) : x > 7$
   $P(3)$ is true.
   $(\exists x)P(x)$ is true.
   For any $x\ in\ E, Q(x)$ is false.
   Hence $(\exists x)P(x) \rightarrow (\exists x)Q(x)$ is false
   $P(6)$ is false and $Q(6)$ is false
   $P(6) \rightarrow Q(6)$ is true.
   $(\exists x)(P(x) \rightarrow Q(x))$ is true.

5. Find the truth value of $(x)(P \rightarrow Q(x)) \vee (x)R(x)$ where $P: 2 > 1, Q(x): x > 3$, $R(x) : x > 4$ with the universe of discourse being $E = \{2,3,4\}$.
   Solution:
   P is true and $Q(4)$ is false, $P \rightarrow Q(4)$ is false
   (x) (P→Q(x)) is false.
   Since $R(2), R(3), R(4)$ are all false.
   $(x)R(x)$ is false.
   Hence $(x)(P \rightarrow Q(x)) \vee (\exists x)R(x)$ is false.

6. Define compound statement function.
   A compound statement function is obtained by combining one or more simple statement functions by logical connectives.
   Eg: $M(x) \wedge H(x), M(x) \rightarrow H(x), M(x) \wedge \neg H(x)$

7. Define Free and Bound variables.
   When a quantifier is used on the variable x or when we assign a value to this variable, we say that this occurrence of the variable is bound. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be free.
   All the variables can be done using a combination of universal quantifiers, existential quantifiers and value assignments.
   Eg: $(x)\ P(x, y)$
   Here $P(x, y)$ is the scope of the quantifier and both occurrence of x are bound occurrences, while the occurrence of y is a free occurrence.

8. Let $P(x)$: x is a person
   $F(x, y)$: x is the father of y
   $M(x, y)$: x is the mother of y.
   Write the predicate " x is the father of the mother of y".
   We symbolize the predicate the name a person called z as the mother of y.
   It is assumed that such a person z exists. We symbolize the predicate as
   $$(\exists z)P(z) \land F(x, z) \land M(z, y)$$

9. Symbolize the expression " All the world loves a mother
   Let $P(x)$: x is a person
   $M(x)$: x is a mother
   $R(x, y)$: x loves y
   The required expression is $(x)(P(x) \rightarrow (y)(P(y) \land L(y) \rightarrow R(x, y))$

10. Symbolize the statement " All men are giants:
    $G(x)$: x is a gaint
    $M(x)$: x is a man
    Symbolically, $(x)\ (M(x) \rightarrow G(x))$

11. Symbolize: For every x, there exists a y such that $x^2 + y^2 \le 100$.
    $$(x)\ (\exists y)\ (x^2 + y^2 \le 100)$$

12. Consider the statement " Give any positive integer, there is a greater positive integer".
    For all x, there exists a y such that y is greater than x. If $G(x, y)$ is " x is greater than y" then the given statement is $(x)(\exists y)\ G(y, x)$
    If we do not impose the restriction on the universe of discourse and if we write $P(x)$ for " x is a positive integer" , then we can symbolize the given statement is
    $$(x)(P(x) \rightarrow (\exists y)(P(y) \land G(y, x)))$$

13. Give the symbolic form of the statement " Every book with a blue cover is a mathematics book"
    Let $B(x)$: x is every book with a blue cover
    $M(x)$: x is mathematics book.
    $(x)\ ((B(x) \rightarrow M(x))$

14. Write each of the following in symbolic form

    i) All men are good.

    ii) No men are good.

Solution:

i) All men are good.

$M(x)$: x is a man

$G(x)$: x is good

$(x)\ [M(x) \rightarrow G(x)]$

ii) No men are good

This can be written as, "For all x, if x is a man, then x is not good"

$$(x)[M(x) \rightarrow \neg G(x)]$$

15. Write each of the following in symbolic form

$i)$*Some men are good.*

$ii)$*Some men are not good.*

Solution:

i)Some men are good.

Let $M(x)$: x is a man

$G(x)$: x is good

$$(\exists x)(M(x) \wedge G(x))$$

ii) Some men are not good.

$$(\exists x)(M(x) \wedge \neg G(x))$$

16. Show that $(x)(H(x) \rightarrow M(x) \wedge H(s)) \Rightarrow M(s)$. Note that this problem is a symbolic translation of a well-known argument known as "Socrates argument" which is given by ,All men are mortal, Socrates is a man, Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal, s : Socrates

We can put the argument in the above form.

$i)\ (x)(H(x) \rightarrow M(x)) \quad Rule\ P$

$ii)\ H(s) \rightarrow M(s) \quad Rule\ US\ From\ i)$

$iii)\ H(s) \quad Rule\ P$

$iv)\ M(s) \quad Rule\ T$

17. Verify the validity of the following argument.

All men are intelligent.

Krishna is a man.

Therefore Krishna is a intelligent.

Solution:

$P(x)$: x is man

$Q(x)$: x is intelligent

S : Krishna

We need to show $(x)(P(x) \rightarrow Q(x)) \wedge P(s) \Rightarrow Q(s)$

$i)\ (x)(P(x) \rightarrow Q(x)) \quad Rule\ P$

$ii) P(s) \rightarrow Q(s) \quad Rule\ US$

$iii)\ P(s) \quad Rule\ P$

$iv) Q(s) \quad Rule\ T\ From\ ii)\ \&\ iii).$

18. Define universe of discourse.
    The variables which are quantified stand for only those objects which are members of a particular set or class. Such a restricted class is called the universe of discourse or the domain of individuals or simply the universe.

19. Consider the statement " Given any positive integer, there is a greater positive integer" . Symbolize this statement with and without using the set of positive integers as the universe of discourse.
    For all x, there exists a y such that y is greater than x. If $G(x, y)$ is " x is greater than y", then the given statement is $(x)(\exists y)\ G(y, x)$.
    If we do not impose the restriction on the universe of discourse and if we writer $P(x)$ for " x is a positive integer", then we can symbolize the given statement is
    $$(x)(P(x) \rightarrow (\exists y)(P(y) \wedge G(y, x)))$$

20. Define Universal quantifiers
    The universal quantification of $P(x)$ is the proposition. "$P(x)$ is true for all values of x in the universe of discourse"
    The notation $(x)P(x)$ denotes the universal quantification of $P(x)$. Here (x) is called the universal quantifier.

21. Define existential quantifier.
    The existential quantification of $P(x)$ is the proposition. "There exists an element x in the universe of discourse such that P(x) is true"
    We use the notation $(\exists x)(P(x))$ for the existential quantification of $P(x)$. Here ∃ is called the existential quantifier.

22. Write the universal specification in quantifiers.
    From $(x)\ A(x)$ one can conclude $A(y)$. If $(x)\ A(x)$ is true for every element x in the universe, then $A(y)$ is true.
    $$(x)A(x) \Rightarrow A(y)$$

23. Define Existential specification in quantifiers
    From $(\exists x)\ A(x)$ one can conclude $A(y)$. If $(\exists x)\ A(x)$ is true for some element x in the universe, then $A(y)$ is true.
    $$(\exists x)A(x) \Rightarrow A(y)$$

24. Define Existential Generalization.
    From $A(x)$ one can conclude $(\exists y)\ A(y)$. If $A(x)$ is true for some element x in the universe, then $(\exists y)\ A(y)$ is true.
    $$A(x) \Rightarrow (\exists y)A(y)$$

25. Define Universal Generalization
    From $A(x)$ one can conclude $(y)A(y)$. If $A(x)$ is true for every element x in the universe, then $(y)\ A(y)$ is true.

$$A(x) \Rightarrow (y)A(y)$$

26. Show that $\neg P(a,b)$ follows logically from $(x)(y)(P(x,y) \rightarrow w(x,y))$ and $\neg w(a,b)$

Solution:

i) $(x)(y)(P(x,y) \rightarrow w(x,y))$ Given premise

ii) $(y)\ P(a,y) \rightarrow w(a,y)$ US

iii) $P(a,b) \rightarrow w(a,b)$ US

iv) $\neg w(a,b)$ Rule P

v) $\neg P(a,b)$ (iii),(iv), Modus tollens

27. If the universe of discourse is finite, then show that $\neg[(\exists x)P(x)] \Leftrightarrow (x)[\neg P(x)]$.

Solution:

Let the universe of discourse be $U = \{x_1, x_2, \ldots, x_n\}$ be finite.

By using DeMorgan's Law of propositional calculus, we have

$$\neg[(\exists x)P(x)] \Leftrightarrow \neg[P(x_1) \vee P(x_2) \ldots \vee P(x_n)]$$
$$\Leftrightarrow \neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n)$$
$$\Leftrightarrow (x)[\neg P(x)]$$

1. i)Show that $(x)((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$

Solution:

1. $(x)((P(x) \rightarrow Q(x)) \wedge (Q(x) \rightarrow R(x))$ *Rule P*
2. $(P(a) \rightarrow Q(a)) \wedge (Q(a) \rightarrow R(a))$ *Rule US* ,1
3. $P(a) \rightarrow Q(a)$ *Rule T* ,2
4. $Q(a) \rightarrow R(a)$ *Rule T* ,2
5. $P(a) \rightarrow R(a)$ *Simplification*, 3,4
6. $(x)(P(x) \rightarrow R(x))$ *Rule UG* ,5

ii)Show that $(\exists x)\ M(x)$ follows logically from the premises.

$(x)(H(x) \rightarrow M(x))$ *and* $(\exists x)H(x)$

Solution:

1. $(x)(H(x) \rightarrow M(x))$ *Rule P*
2. $H(a) \rightarrow M(a)$ *Rule US* ,1
3. $(\exists x)H(x)$ *Rule P*
4. $H(a)$ *Rule ES*, 3
5. $M(a)$ *Rule Modus ponens* ,2,4
6. $(\exists x)M(x)$ *Rule EG* ,5

2. i) Prove that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

Solution:

1. $(\exists x)(P(x) \wedge Q(x))$ *Rule P*
2. $P(a) \wedge Q(a)$ *Rule ES* ,1
3. $P(a)$ *Rule T*, 2
4. $(\exists x)P(x)$ *Rule EG*, 3
5. $Q(a)$ *Rule T*, 2
6. $(\exists x)Q(x)$ *Rule EG* ,5
7. $(\exists x)P(x) \wedge (\exists x)Q(x)$ *Rule Conjuction* ,4,6

ii) Show that from $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow w(y))$,

$$(\exists y)(M(y) \wedge \neg w(y)) \Rightarrow (x)F(x) \rightarrow \neg S(x)$$

Solution:

1. $(\exists y)(M(y) \wedge \neg w(y))$ — *Rule P*
2. $M(a) \wedge \neg w(a)$ — *Rule ES* ,1
3. $\neg(\neg M(a) \vee w(a))$ — 2, *De Morgan's law*
4. $\neg\big(M(a) \rightarrow w(a)\big)$ — 3, *disjunction as conditional*
5. $(\exists x)\big(F(x) \wedge S(x)\big) \rightarrow (y)\big(M(y) \rightarrow w(y)\big)$ — *Rule P*
6. $\big(F(b) \wedge S(b)\big) \rightarrow \big(M(a) \rightarrow w(a)\big)$ — *Rule ES* ,5
7. $\neg\big(F(b) \wedge S(b)\big)$ — *Modus tollens* ,4,6
8. $\neg F(b) \vee \neg S(b)$ — 7, *De Morgan's law*
9. $F(b) \rightarrow \neg S(b)$ — 8, *disjunction as conditional*
10. $(x)F(x) \rightarrow \neg S(x)$ — 9, *UG*

3. i) Show that $(x)\ (P(x) \vee (Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Solution:

Let us prove this by indirect method

Let us assume that $\neg\big((x)P(x) \vee (\exists x)Q(x)\big)$ as additional premise

1. $\neg\big((x)P(x) \vee (\exists x)Q(x)\big)$ — *Additional premise*
2. $\neg(x)\ P(x) \wedge \neg\ (\exists x)Q(x)$ — 1, *De Morgan's law*
3. $\neg(x)\ P(x)$ — *Rule T*, 2
4. $(\exists x)\neg\ P(x)$ — 3, *De Morgan's law*
5. $\neg\ P(a)$ — *Rule ES*, 4
6. $\neg\ (\exists x)Q(x)$ — *Rule T*, 2
7. $(x)\neg\ Q(x)$ — 6, *De Morgan's law*
8. $\neg Q(a)$ — *Rule US*, 7
9. $\neg\ P(a) \wedge \neg Q(a)$ — 5,8, *conjunction*
10. $\neg\ \big(P(a) \vee Q(a)\big)$ — 9, *De Morgan's law*
11. $(x)\ (P(x) \vee (Q(x))$ — *Rule P*
12. $P(a) \vee Q(a)$ — *Rule US*, 11
13. $\neg\ \big(P(a) \vee Q(a)\big) \wedge \big(P(a) \vee Q(a)\big)$ — 11,12, *conjunction*
14. F — *Rule T*, 13

ii) There is mistake in the following derivation. Find it. Is the conclusion valid?. If so, obtain a correct derivation.

1. $(x)(P(x) \rightarrow Q(x))$ — *Rule P*
2. $P(y) \rightarrow Q(y)$ — *US*
3. $(\exists x)P(x)$ — *Rule P*
4. $P(y)$ — *ES*
5. $Q(y)$ — *Rule T*, 2,4
6. $(\exists x)Q(x)$ — *EG*

Solution:

1. $(x)(P(x) \rightarrow Q(x))$ — *Rule P*
2. $P(a) \rightarrow Q(a)$ — *Rule US* ,2
3. $(\exists y)P(y)$ — *Rule P*
4. $P(a)$ — *Rule ES* ,3
5. $Q(a)$ — *T*, 2,4, *and modus ponens*
6. $(\exists z)Q(z)$ — *Rule EG* ,5

Therefore $(\exists z)Q(z)$ is validly derivable from the premises
$(x)(P(x) \rightarrow Q(x)), (\exists y)P(y)$

4.i) Obtain the following implication by indirect method.
$(x)(P(x) \rightarrow Q(x)), (x)(R(x) \rightarrow \neg Q(x)) \Rightarrow (x)(R(x) \rightarrow \neg P(x))$

Solution:

Contrapositive method:

Let us assume that $\neg(x)(R(x) \rightarrow \neg P(x))$ as additional premise.

1. $\neg(x)(R(x) \rightarrow \neg P(x))$ $\quad$ *Rule additional P*
2. $(\exists x)\neg(R(x) \rightarrow \neg P(x))$ $\quad$ Demorgan's law ,1
3. $\neg(R(a) \rightarrow \neg P(a))$ $\quad$ $2, ES$
4. $\neg(\neg R(a) \vee \neg P(a))$ $\quad$ $T, 2,$ *and equivalence*
5. $R(a) \wedge P(a)$ $\quad$ Demorgan's law ,4
6. $R(a)$ $\quad$ $T, 5$
7. $P(a)$ $\quad$ $T, 5$
8. $(x)(R(x) \rightarrow \neg Q(x))$ $\quad$ *Rule P*
9. $R(a) \rightarrow \neg Q(a)$ $\quad$ *Rule US* ,3
10. $\neg Q(a)$ $\quad$ $T, 6{,}9,$ *and modus ponens*
11. $P(a) \wedge \neg Q(a)$ $\quad$ $T, 7{,}10,$ *and* conjuction
12. $\neg(\neg P(a) \vee Q(a))$ $\quad$ $T, 6,$ Demorgan's law
13. $\neg(P(a) \rightarrow Q(a))$ $\quad$ $T, 12,$ *and equivalence*
14. $(x)(P(x) \rightarrow Q(x))$ $\quad$ *Rule P*
15. $P(a) \rightarrow Q(a)$ $\quad$ *Rule US* ,14
16. $(\neg(P(a) \rightarrow Q(a))) \wedge P(a) \rightarrow Q(a)$ $\quad$ $T, 13{,}15$ *and conjuction*
17. F $\quad$ $T, 16$ *and negation law*

ii) Is the following conclusion validly derivable from the premises given?

$$(x)(P(x) \rightarrow Q(x)), (\exists y)P(y) \Rightarrow (\exists z)Q(z).$$

Solution:

1. $(x)(P(x) \rightarrow Q(x))$ $\quad$ *Rule P*
2. $P(a) \rightarrow Q(a)$ $\quad$ *Rule US* ,2
3. $(\exists y)P(y)$ $\quad$ *Rule P*
4. $P(a)$ $\quad$ *Rule ES* ,3
5. $Q(a)$ $\quad$ $T, 2{,}4,$ *and modus ponens*
6. $(\exists z)Q(z)$ $\quad$ *Rule EG* ,5

Therefore $(\exists z)Q(z)$ is validly derivable from the premises
$(x)(P(x) \rightarrow Q(x)), (\exists y)P(y)$

5.i) Use indirect method of proof show that
$(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

Solution:

Let us assume that $\neg((\exists x)A(x) \vee (\exists x)B(x))$ as additional premise

1. $\neg((\exists x)A(x) \vee (\exists x)B(x))$ $\quad$ *Additional premise*
2. $\neg(\exists x)A(x) \wedge \neg(\exists x)B(x)$ $\quad$ $1,$ *De Morgan's law*
3. $(x)\neg A(x) \wedge (x)\neg B(x)$ $\quad$ $2,$ *De Morgan's law*
4. $\neg A(a) \wedge \neg B(a)$ $\quad$ *Rule US*, 3
5. $\neg(A(a) \vee B(a))$ $\quad$ $4,$ *De Morgan's law*

6. $(\exists x)(A(x) \vee B(x))$ — $Rule\ P$
7. $A(a) \vee B(a)$ — $Rule\ ES, 6$
8. $\neg\ (A(a) \vee B(a)) \wedge (A(a) \vee B(a))$ — $5,7, conjunction$
9. F — $Rule\ T, 8\ and\ negation\ law$

ii) Obtain the following implication.

$(x)(P(x) \to Q(x)), (x)(R(x) \to \neg Q(x)) \Rightarrow (x)(R(x) \to \neg P(x))$

Solution:

1. $(x)(P(x) \to Q(x))$ — $Rule\ P$
2. $P(a) \to Q(a)$ — $Rule\ US\ ,1$
3. $(x)(R(x) \to \neg Q(x))$ — $Rule\ P$
4. $R(a) \to \neg Q(a)$ — $Rule\ US\ ,3$
5. $Q(a) \to \neg R(a)$ — $T, 4, and\ equivalence$
6. $P(a) \to \neg R(a)$ — $T, 2,5, and$ hypothetical syllogism
7. $R(a) \to \neg P(a)$ — $T, 6, and\ equivalence$
8. $(x)(R(x) \to \neg P(x))$ — $7, UG$

6.i) Prove that $(\exists x)(P(x) \wedge S(x)), (x)\ (P(x) \to R(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$

Solution:

1. $(x)(P(x) \to R(x))$ — $Rule\ P$
2. $P(a) \to R(a)$ — $Rule\ US\ ,2$
3. $(\exists x)(P(x) \wedge S(x))$ — $Rule\ P$
4. $P(a) \wedge S(a)$ — $Rule\ ES\ ,3$
5. $P(a)$ — $T, 4, and\ conjuction$
6. $S(a)$ — $T, 4, and\ conjuction$
7. $R(a)$ — $T, 2,5, and\ modus\ ponens$
8. $R(a) \wedge S(a)$ — $T, 6,7\ and\ conjuction$
9. $(\exists x)\big(R(x) \wedge S(x)\big)$ — $EG, 8$

ii) By indirect method prove that $(x)\ (P(x) \to Q(x)), (\exists x)P(x) \Rightarrow (\exists x)Q(x)$

Solution:

Let us assume that $\neg(\exists x)Q(x)$ as additional premise

1. $\neg(\exists x)Q(x)$ — $Additional\ premise$
2. $(x)\neg Q(x)$ — $1, De\ Morgan's\ law$
3. $\neg Q(a)$ — $Rule\ US, 2$
4. $(\exists x)P(x)$ — $Rule\ P$
5. $P(a)$ — $Rule\ ES, 4$
6. $P(a) \wedge \neg Q(a)$ — $5,3\ and\ conjuction$
7. $\neg\big(\neg P(a) \vee Q(a)\big)$ — $6, De\ Morgan's\ law$
8. $\neg\big(P(a) \to Q(a)\big)$ — $T, 7, Equivalence$
9. $(x)\big(P(x) \to Q(x)\big)$ — $Rule\ P$
10. $P(a) \to Q(a)$ — $Rule\ US, 9$
11. $\neg\big(P(a) \to Q(a)\big) \wedge P(a) \to Q(a)$ — $8,10\ and\ conjuction$
12. F — $Rule\ T, 11\ and\ negation\ law$

7.i) $(x)(H(x) \to A(x)) \Rightarrow (x)\ ((\exists y)(H(y) \wedge N(x,y)) \to (\exists y)(A(y) \wedge N(x,y)))$

Solution:

Let us assume that $\neg(x)\ ((\exists y)(H(y) \wedge N(x,y)) \to (\exists y)(A(y) \wedge N(x,y)))$ as additional premise.

1. $\neg(x)\ ((\exists y)(H(y) \wedge N(x,y)) \rightarrow\ (\exists y)(A(y) \wedge N(x,y)))$ Additional premise
2. $(\exists x)\neg\ ((\exists y)(H(y) \wedge N(x,y)) \rightarrow\ (\exists y)(A(y) \wedge N(x,y)))$ $1, De\ Morgan's\ law$
3. $\neg\ ((\exists y)(H(y) \wedge N(a,y)) \rightarrow\ (\exists y)(A(y) \wedge N(a,y)))$ $2, ES$
4. $\neg\ (\neg(\exists y)(H(y) \wedge N(a,y)) \vee (\exists y)(A(y) \wedge N(a,y)))$ $T, 3, Equivalence$
5. $(\exists y)(H(y) \wedge N(a,y)) \wedge \neg(\exists y)(A(y) \wedge N(a,y))$ $4, De\ Morgan's\ law$
6. $(\exists y)\big(H(y) \wedge N(a,y)\big)$ $T, 5, conjuction$
7. $H(b) \wedge N(a,b)$ $ES, 6$
8. $H(b)$ $T, 7, conjuction$
9. $N(a,b)$ $T, 7, conjuction$
10. $\neg(\exists y)(A(y) \wedge N(a,y))$ $T, 5, conjuction$
11. $(y)(\neg A(y) \vee \neg N(a,y))$ $10, De\ Morgan's\ law$
12. $(\neg A(b) \vee \neg N(a,b))$ $11, US$
13. $A(b) \rightarrow\ \neg N(a,b)$ $T, 12, Equivalence$
14. $\neg A(b)$ $T, 9,13, Modus\ tollens$
15. $H(b) \wedge \neg A(b)$ $T, 8,14, conjuction$
16. $\neg\big(\ \neg H(b) \vee A(b)\big)$ $15, De\ Morgan's\ law$
17. $\neg\big(\ H(b) \rightarrow A(b)\big)$ $T, 16, Equivalence$
18. $(x)\big(H(x) \rightarrow A(x)\big)$ $Rule\ P$
19. $H(b) \rightarrow A(b)$ $US, 18$
20. $\neg\big(\ H(b) \rightarrow A(b)\big) \wedge \big(H(b) \rightarrow A(b)\big)$ $T, 17,19\ and\ conjuction$
21. F $Rule\ T, 20\ and\ negation\ law$

ii) Prove that $(\exists x)A(x) \rightarrow (x)B(x)\ \Rightarrow\ (x)(A(x) \rightarrow B(x))$

Solution:

Let us assume that $\neg(x)(A(x) \rightarrow B(x))$ as additional premise.

1. $\neg(x)\big(A(x) \rightarrow B(x)\big)$ Additional premise
2. $(\exists x)\neg\ \big(A(x) \rightarrow B(x)\big)$ $1, De\ Morgan's\ law$
3. $\neg\ \big(A(a) \rightarrow B(a)\big)$ $2, ES$
4. $(\exists x)A(x) \rightarrow (x)B(x)$ $Rule\ P$
5. $A(a) \rightarrow B(a)$ $4, ES$
6. $\neg\ \big(A(a) \rightarrow B(a)\big) \wedge A(a) \rightarrow B(a)$ $T, 3, 5\ and\ conjuction$
7. F $Rule\ T, 6\ and\ negation\ law$

8.i) Using CP , obtain the following implication.

$(x)(P(x) \rightarrow Q(x)), (x)(R(x) \rightarrow\ \neg Q(x))\ \Rightarrow\ (x)(R(x) \rightarrow\ \neg P(x))$

Solution:

1. $(x)(P(x) \rightarrow Q(x))$ $Rule\ \ P$
2. $P(a) \rightarrow Q(a)$ $2, US$
3. $(x)(R(x) \rightarrow\ \neg Q(x))$ $Rule\ \ P$
4. $R(a) \rightarrow\ \neg Q(a)$ $Rule\ US$,3
5. $Q(a) \rightarrow\ \neg R(a)$ Rule CP ,4
6. $P(a) \rightarrow\ \neg R(a)$ $T, 2,5, hypothetical\ syllogism$
7. $R(a) \rightarrow\ \neg P(a)$ Rule CP ,6
8. $(x)\big(R(x) \rightarrow\ \neg P(x)\big)$ $Rule\ UG, 7$