Unit-III

Set Theory and Boolean Algebra

Part A

2 Marks Questions

1. ***For any sets A,B and C prove that*** $\boldsymbol{A\times(B\cap C) = (A \times B)\cap(A \times C)}$

   Solution:

   Let $(x, y) \in A\times (B \cap C)$

   $x \in A \, and \, y \in (B \cap C)$

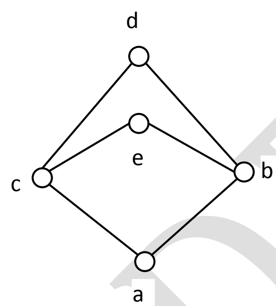
   $(x \in A \, and \, y \in B) \, and \, (x \in A \, and \, y \in c)$

   $(x, y) \in A \times B \, and \, (x, y) \in A\times C$

   $(x , y) \in (A \times B) \cap (A \times C)$

   Hence $A\times(B\cap C) = (A \times B)\cap(A \times C)$

2. ***The following is the hasse diagram of a partially ordered set. Verify whether it is a lattice.***



   Solution:

   d and e are the upper bounds of c and b. As d and e cannot be compared, therefore the $LUB\,\{c, b\}$ does not exists. The Hasse diagram is not a lattice.

3. ***Give an example of a relation which is symmetric, transitive but not reflexive on*** $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$

   Solution:

   $R = \{(a, a), (a, b), (b, a), (b, b)\}$

4. ***Define partially ordered set.***

   A Set with a partially ordering relation is called a poset or partially ordered set.

5. ***Find the Partition of*** $\boldsymbol{A = \{0, 1, 2, 3, 4, 5\}}$ ***with minsets generated by*** $\boldsymbol{B_1 = \{0, 2, 4\}}$ ***and*** $\boldsymbol{B_2 = \{1, 5\}}$.

   Solution:

   $B_1 \cap B_2 = \emptyset, B_1 \cup B_2 = \{0, 1, 2, 4, 5\} \neq A \, , \, (B_1 \cup B_2)' = \{3\}$

   $B_1 \cup B_2 \cup (B_1 \cap B_2)' = \{0, 1, 2, 3, 4, 5\} = A$

   Partition of $A = \{\{0, 2, 4\}, \{1, 5\}, \{3\}\}$

6. ***If a poset has a least element, then prove it is unique.***

   Proof:

   Let $\langle \boldsymbol{L}, \leq \rangle$ be a poset with a_1, a_2 be two least elements.

   If a_1 is the least element, $a_1 \leq a_2$

   If a_2 is the least element $a_2 \leq a_1$

   By antisymmetric property $a_1 = a_2$

   So that least element is unique.

7. ***If $R = \{(1,1),(1,2),(2,3)\}$ and $S = \{(2,1),(2,2),(3,2)\}$ are the relations on the set $A = \{1,2,3\}$. Verify whether $RoS = SoR$ by finding the relation matrices of RoS and SoR.***

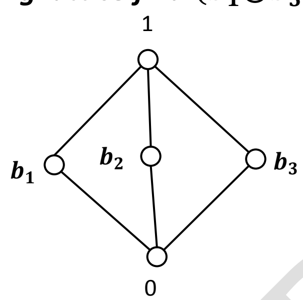
Solution:

$$M_R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, M_s = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{RoS} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } M_{SoR} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$M_{RoS} \neq M_{SoR} \Rightarrow RoS \neq SoR$

8. ***In the following lattice find $(b_1 \oplus b_3) * b_2$***



Solution:

$b_1 \oplus b_3 = 1$. Hence $(b_1 \oplus b_3) * b_2 = 1 * b_2 = b_2$

9. ***If $A_2 = \{\{1,2\},\{3\}\}, A_2 = \{\{1\},\{2,3\}\}$ and $A_3 = \{\{1,2,3\}\}$ then show that A_1, A_2 and A_3 are mutually disjoint.***

Solution:

$A_1 \cap A_2 = \emptyset, A_1 \cap A_3 = \emptyset, A_2 \cap A_3 = \emptyset$

Hence A_1, A_2 *and* A_3 are mutually disjoint.

10. ***Let $x = \{1,2,3,4\}$. If***

$\boldsymbol{R = \{< x, y > \mid x \in X \wedge y \in X \wedge (x - y) \text{is an nonzero multiple of } 2\}}$

$\boldsymbol{S = \{< x, y > \mid x \in X \wedge y \in X \wedge (x - y) \text{is an nonzero multiple of } 3\}}$

Find $R \cup S$ and $R \cap S$.

Solution:

$R = \{(1,3),(3,1),(2,4),(4,2)\}, S = \{(1,4),(4,1)\}$

$R \cup S = \{(1,3),(3,1),(2,4),(4,2),(1,4),(4,1)\}, R \cap S = \emptyset$

$R \cap S = \{< x, y > \mid x \in X \wedge y \in X \wedge (x - y) \text{is an nonzero multiple of } 6\}$

11. ***If R and S are reflexive relations on a set A, then show that $R \cup S$ and $R \cap S$ are also reflexive relations on A.***

Solution:

Let $a \in A$. Since R *and* S are reflexive.

We have $(a,a) \in R$ *and* $(a,a) \in S \Rightarrow (a,a) \in R \cap S$

Hence $R \cap S$ is reflexive.

$(a,a) \in R$ *or* $(a,a) \in S \Rightarrow (a,a) \in R \cup S$

Hence $R \cup S$ is reflexive.

12. Define Equivalence relation. Give an example

Solution:

A relation R in a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

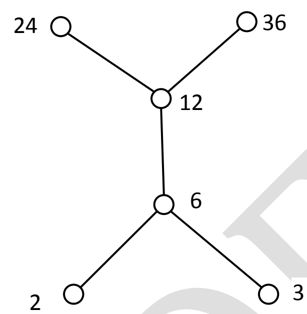
Eg: i) Equality of numbers on a set of real numbers

ii) Relation of lines being parallel on a set of lines in a plane.

13. Let $X = \{2, 3, 6, 12, 24, 36\}$ and the relation be such that $x \leq y \ iff$ $x \ divides \ y$. Draw the Hasse Diagram of $\langle X, \leq \rangle$.

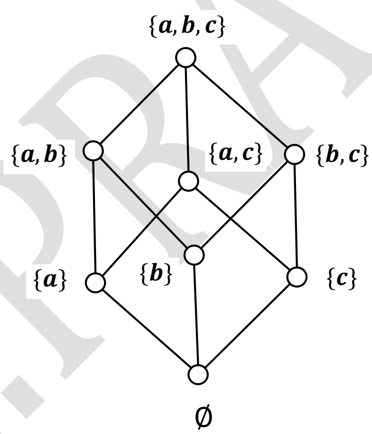
Solution:

The Hasse diagram is



14. Let A be a given finite set and $P(A)$ its power set. Let $\subseteq$ be the inclusion relation on the elements of $P(A)$. Draw Hasse diagram of $\langle P(A), \leq \rangle$ for $A = \{a, b, c\}$

Solution:



15. Verify $B \cup (\cap_{i \in I} A_i) = \cap_{i \in I} (B \cup A_i)$. If $A_1 = \{1, 5\}, A_2 = \{1, 2, 4, 6\}$, $A_3 = \{3, 4, 7\}, B = \{2, 4\}$ and $I = \{1, 2, 3\}$

Solution:

$\cap_{i \in I} A_i = A_1 \cap A_2 \cap A_3 = \emptyset$

$B \cup (\cap_{i \in I} A_i) = \{2,4\} \ldots (1)$

$B \cup A_1 = \{1, 2, 4, 5\}, B \cup A_2 = \{1, 2, 4, 6\}, B \cup A_3 = \{2, 3, 4, 7\}$

$\cap_{i \in I} (B \cup A_i) = (B \cup A_1) \cap (B \cup A_2) \cap (B \cup A_3) = \{2,4\} \ldots (2)$

from (1) and (2) we get

$B \cup (\cap_{i \in I} A_i) = \cap_{i \in I} (B \cup A_i)$

16. If $\boldsymbol{A = \{1, 2, 3\}, B = \{a, b\}}$ find $\boldsymbol{A \times B}$ and $\boldsymbol{B \times A}$ and prove that $\boldsymbol{n(A \times B) = n(B \times A)}$

Solution:

$A \times B = \{(1, a), (2, a), (3, a), (1, b), (2, b), (3, b)\}, n(A \times B) = 6$

$B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}, n(B \times A) = 6$

$$\therefore n(A \times B) = n(B \times A)$$

17. ***Show that $\boldsymbol{(A \cap B)' = A' \cup B'}$***

Proof:

Let $x \in (A \cap B)' \Leftrightarrow x \notin (A \cap B)$

$\Leftrightarrow x \notin A \text{ or } x \notin B$

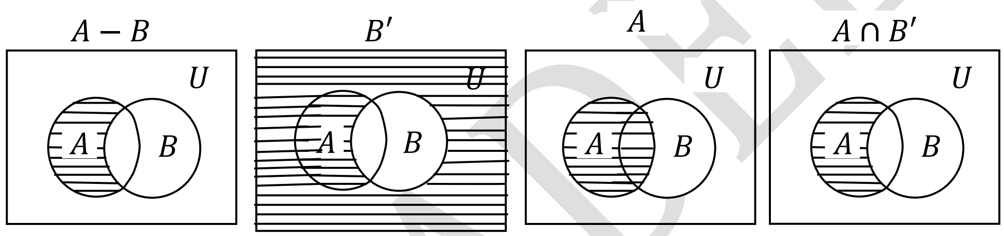
$\Leftrightarrow x \in A' \text{ or } x \in B'$

$\Leftrightarrow x \in A' \cup B'$

Hence $(A \cap B)' = A' \cup B'$

18. Draw venn diagram and prove $\boldsymbol{A - B = A \cap B'}$

Solution:



$\therefore A - B = A \cap B'$

19. Find x and y given $\boldsymbol{(2x, x + y) = (6, 2)}$

Solution:

Given two ordered pairs are equal if and only if corresponding components are equal.

$2x = 6 \quad \Rightarrow x = 3$

$x + y = 2 \Rightarrow 3 + y = 2 \Rightarrow y = -1$

20. Write the representing each of the relations from $\begin{pmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{pmatrix}$

Solution:

Let $A = \{1, 2, 3\}$ and R be the relation defined on A corresponding to the given matrix. $\therefore R = \{(1,1), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2), (3,3)\}$

21. Which elements of the poset $\boldsymbol{[\{2, 4, 5, 10, 12, 20, 25\}, /\ ]}$ are maximal and which are minimal?

(or)

Give an example for a poset that have more than one maximal element and more than one minimal element.

Solution:

$A = [\{2, 4, 5, 10, 12, 20, 25\}, \ /\ ], / \textit{ is the division relation.}$

The maximal elements are 12, 20, 25 and the minimal elements are 2,5 .

22. Define Lattice

A Lattice in a partially ordered set $\langle \boldsymbol{L}, \leq \rangle$ in which every pair of elements $a, b \in L$ has the greatest lower bound and a least upper bound.

23. Let $\langle \boldsymbol{L}, \leq \rangle$ be a lattice. For any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{L}$ we have $\boldsymbol{a} * \boldsymbol{a} = \boldsymbol{a}$

Solution:

Since $a \leq a$, a is a lower bound of $\{a\}$. If b is any lower bound of $\{a\}$, then we have $b \leq a$. Thus we have $a \leq a$ or $b \leq a$ equivalently, a is an lower bound for $\{a\}$and any other lower bound of $\{a\}$ is smaller than a. This shows that a is the greatest lower bound of $\{a\}$, i.e., $GLB\{a\} = a$

$\therefore a * a = GLB\{a\} = a$

24. Define sublattice

Let $\langle L, *, \oplus \rangle$ be a lattice and let $S \subseteq L$ be a subset of L. Then $\langle S, *, \oplus \rangle$ is a sublattice of $\langle L, *, \oplus \rangle$ iff S is closed under both operations $*$ and $\oplus$.

25. Define Lattice Homomorphism

Let $\langle L, *, \oplus \rangle$ and $\langle S, \wedge, \vee \rangle$ be two lattices. A mapping $g: L \to S$ is called a lattice homomorphism from the lattice $\langle L, *, \oplus \rangle$to $\langle S, \wedge, \vee \rangle$ if for any $a, b \in L$

$g(a * b) = g(a) \wedge g(b) \; and \; g(a \oplus b) = g(a) \vee g(b)$

26. Define Modular

A lattice $\langle L, *, \oplus \rangle$ is called modular if for all $x, y, z \in L$

$$x \leq z \Rightarrow x \oplus (y * z) = (x \oplus y) * z$$

27. ***Define Distributive lattice***.

A Lattice $\langle L, *, \oplus \rangle$is called a distributive lattice if for any $a, b, c \in L$

$a * (b \oplus c) = (a * b) \oplus (a * c)$

$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

28. Prove that every distributive lattice is modular.

Proof:

Let $\langle L, *, \oplus \rangle$ be a distributive lattice.

$\forall a, b, c \in L$ we have , $a \oplus (b * c) = (a \oplus b) * (a \oplus c) \ldots (1)$

Thus if $a \leq c$ then $a \oplus c = c \ldots (2)$

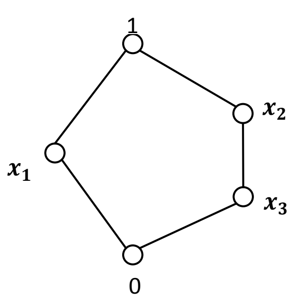
from (1) and (2) we get

$a \oplus (b * c) = (a \oplus b) * c$

So if $a * c$,then $a \oplus (b * c) = (a \oplus b) * c$.

$\therefore L$ is modular.

29. The lattice with the following Hasse diagram is not distributive and not modular.



Solution:

In this case, $(x_1 \oplus x_3) * x_2 = 1 * x_2 = x_2 \ldots (1)$
And $(x_1 * x_2) \oplus (x_3 * x_2) = 0 \oplus x_3 = x_3 \ldots (2)$
From (1) and (2) we get
$(x_1 \oplus x_3) * x_2 \neq (x_1 * x_2) \oplus (x_3 * x_2)$
Hence the lattice is not distributive.
$x_3 < x_2 \Rightarrow x_3 \oplus (x_1 * x_2) = x_3 \oplus 0 = x_3 \ldots (3)$
$(x_3 \oplus x_1) * x_2 = 1 * x_2 = x_2 \ldots (4)$
From (3) and (4) we get
$x_3 \oplus (x_1 * x_2) \neq (x_3 \oplus x_1) * x_2$
Hence the lattice is not modular.

30. Prove that $A \subset B \Leftrightarrow A \cap B = A$

Proof:

i) Given $A \subset B$.
Let $x \in A \cap B$
$\Rightarrow x \in A \ and\ x \in B$
$\Rightarrow x \in A$ (In particular)
$\therefore A \cap B \subset A \ldots (1)$
$Let\ x \in A \Rightarrow x \in A\ and\ x \in B$
$\Rightarrow x \in A \cap B$
$\therefore A \subset A \cap B \ldots (2)$
From (1) and (2) we get
$A \subset B \Rightarrow A = A \cap B$

ii)Converse:
Let $A = A \cap B$ to prove $A \subset B$
Let $x \in A \Rightarrow x \in A \cap B$
$\Rightarrow x \in A\ and\ x \in B$
$\Rightarrow x \in B$ (In particular)
$\therefore A \subset B$
From (i) and (ii) we get
$\therefore A \subset B \Leftrightarrow A \cap B = A$

PART-B

1. i) Prove that distinct equivalence classes are disjoint.
   Solution:
   Let R be an equivalence relation defined on set X.
   Let $[x]_R, [y]_R$ are two distinct equivalence classes on X
   i.e., $x \not R y$
   Let us assume that there is at least one element $z \in [x]_R\ and\ also\ z \in [y]_R$
   i.e., $xRz\ and\ yRz \Rightarrow zRy (By\ symmetric)$
   $\therefore xRz\ and\ zRy \Rightarrow xRy (By\ transitivity)$

Which is a contradiction.

$$[x]_R \cap [y]_R = \emptyset$$

∴Distinct equivalence classes are disjoint.

ii) In a Lattice, show that $a = b \text{ and } c = d \Rightarrow a * c = b * d$

Solution:

For any $a, b, c \in L$

If $a = b \Rightarrow c * a \leq c * b$

$$\Rightarrow a * c \leq b * c \ldots (1)(By\ Commutative\ law)$$

For any $b, c, d \in L$

If $c = d \Rightarrow b * c \leq b * d \ldots (2)$

From (1) and (2) we get

$$a * c = b * d$$

iii) In a distributive Lattice prove that

$a * b = a * c \text{ and } a \oplus b = a \oplus c \Rightarrow b = c.$

Solution:

$$(a * b) \oplus c = (a * c) \oplus c = c \ldots (1)\ [a * b = a * c \text{ and absorbtion law}]$$

$$(a * b) \oplus c = (a \oplus c) * (b \oplus c)\ [Distributive\ law]$$

$$= (a \oplus b) * (b \oplus c) = (a \oplus b) * (c \oplus b)\ [a \oplus b = a \oplus c \text{ and commutative law}]$$

$$= (a * c) \oplus b = (a * b) \oplus b = b \ldots (2)[Distributive\ and\ absorbtion\ law]$$

From (1) and (2) we get,

$$b = c$$

2. i) Let $P = \{\{1,2\}, \{3,4\}, \{5\}\}$ be a partition of the set $S = \{1,2,3,4,5\}$. Construct an equivalence relation R on S so that the equivalence classes with respect to R are precisely the members of P.

Solution:

Let $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$

Since $(1,1), (2,2), (3,3), (4,4), (5,5) \in R$

$$\therefore R \text{ is reflexive}$$

$$For\ (1,2), (3,4) \in R \text{ there is } (2,1), (4,3) \in R$$

$$\therefore R \text{ is Symmetric}$$

$$For\ (1,2) \text{ and } (2,1) \in R \text{ there is } (1,1) \in R$$

$$For\ (2,1) \text{ and } (1,2) \in R \text{ there is } (2,2) \in R$$

$$For\ (3,4) \text{ and } (4,3) \in R \text{ there is } (3,3) \in R$$

$$For\ (4,3) \text{ and } (3,4) \in R \text{ there is } (4,4) \in R$$

$$\therefore R \text{ is transitive}$$

$$\therefore R \text{ is an equivalence relation}$$

$$[1]_R = \{1,2\}, [3]_R = \{3,4\}, [5]_R = \{5\}$$

Equivalence classes with respect to $R = \{[1]_R, [3]_R, [5]_R\}$

The equivalence classes with respect to R are precisely the members of P

ii) Establish De Morgan's laws in a Boolean algebra

Solution: Let $a, b \in (B, *, \oplus, ', 0, 1)$

To prove $(a \oplus b)' = a' * b'$

$$(a \oplus b) * (a' * b') = \big(a * (a' * b')\big) \oplus \big(b * (a' * b')\big)$$
$$= \big(a * (a' * b')\big) \oplus \big((a' * b') * b\big)$$
$$= \big((a * a') * b'\big) \oplus \big(a' * (b' * b)\big)$$
$$= (0 * b') \oplus (a' * 0) = 0 \oplus 0$$
$$(a \oplus b) * (a' * b') = 0 \ldots (1)$$
$$(a \oplus b) \oplus (a' * b') = \big((a \oplus b) \oplus a'\big) * \big((a \oplus b) \oplus b'\big)$$
$$= \big((b \oplus a) \oplus a'\big) * \big((a \oplus b) \oplus b'\big)$$
$$= \big(b \oplus (a \oplus a')\big) * \big(a \oplus (b \oplus b')\big)$$
$$= (b \oplus 1) * (a \oplus 1) = 1 * 1$$
$$(a \oplus b) \oplus (a' * b') = 1 \ldots (2)$$

From (1) *and* (2) *we get*,

$$\therefore (a \oplus b)' = a' * b'$$

To prove $(a * b)' = a' \oplus b'$

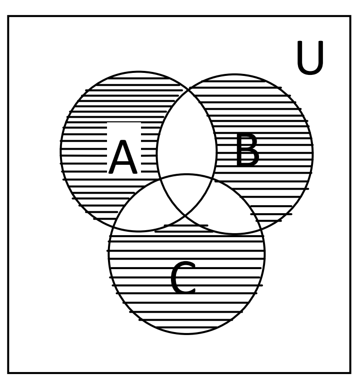
$$(a * b) \oplus (a' \oplus b') = \big(a \oplus (a' \oplus b')\big) * \big(b \oplus (a' \oplus b')\big)$$
$$= \big(a \oplus (a' \oplus b')\big) * \big((a' \oplus b') \oplus b\big)$$
$$= \big((a \oplus a') \oplus b'\big) * \big(a' \oplus (b' \oplus b)\big)$$
$$= (1 \oplus b') * (a' \oplus 1) = 1 * 1$$
$$(a * b) \oplus (a' \oplus b') = 1 \ldots (3)$$
$$(a * b) * (a' \oplus b') = \big((a * b) * a'\big) \oplus \big((a * b) * b'\big)$$
$$= \big((b * a) * a'\big) \oplus \big((a * b) * b'\big)$$
$$= \big(b * (a * a')\big) \oplus \big(a * (b * b')\big)$$
$$= (b * 0) \oplus (a * 0) = 0 \oplus 0$$
$$(a * b) * (a' \oplus b') = 0 \ldots (4)$$

From (3) *and* (4) *we get*,

$$(a * b)' = a' \oplus b'$$

3. i) A survey of 500 television watches produced the following information. 285 watch football games; 195 watch hockey games, 115 watch Basket ball games; 45 watch football and basket ball games; 70 watch football and hockey games; 50 watch hockey and basket ball games; 50 do not watch any of the three games. How many people watch exactly one of the three games?

Solution:



Let U denote the television watchers
Let A denote the football game watchers
Let B denote the hockey game watchers
Let C denote the basketball game watchers

$|U| = 500, |A| = 285, |B| = 195, |C| = 115, |A \cap B| = 70, |A \cap C| = 45,$
$|B \cap C| = 50, |(A \cup B \cup C)'| = 50$

$\textit{The shaded portion in the above venn diagram gives the number of people watch exactly one of the three games.}$

$|(A \cup B \cup C)| = |U| - |(A \cup B \cup C)'| = 500 - 50 = 450$

$\textit{The number of people watch all three games} = |A \cap B \cap C|$

$\textit{We know that}$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$450 = 285 + 195 + 115 - 70 - 45 - 50 + |A \cap B \cap C|$

$|A \cap B \cap C| = 20.$

$$\left.\begin{array}{c}\textit{The number of people}\\ \textit{watch football only}\end{array}\right\} = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= 285 - 70 - 45 + 20 = 190 \ldots (a)$$

$$\left.\begin{array}{c}\textit{The number of people}\\ \textit{watch hockey only}\end{array}\right\} = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$
$$= 195 - 70 - 50 + 20 = 95 \ldots (b)$$

$$\left.\begin{array}{c}\textit{The number of people}\\ \textit{watch basketball only}\end{array}\right\} = |C| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$
$$= 115 - 50 - 45 + 20 = 40 \ldots (c)$$

$$\left.\begin{array}{c}\textit{The number of people watch exactly one}\\ \textit{of the three games}\end{array}\right\} = (a) + (b) + (c)$$
$$= 190 + 95 + 40 = 325$$

ii) In a Boolean algebra L, Prove that $(a \wedge b)' = a' \vee b', \forall\, a, b \in L$

Solution:

$$(a \wedge b) \vee (a' \vee b') = \big(a \vee (a' \vee b')\big) \wedge \big(b \vee (a' \vee b')\big)$$
$$= \big(a \vee (a' \vee b')\big) \wedge \big((a' \vee b') \vee b\big)$$
$$= \big((a \vee a') \vee b'\big) \wedge \big(a' \vee (b' \vee b)\big)$$
$$= (1 \vee b') \wedge (a' \vee 1) = 1 * 1$$
$$(a * b) \vee (a' \vee b') = 1 \ldots (1)$$
$$(a \wedge b) \wedge (a' \vee b') = \big((a \wedge b) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big((b \wedge a) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big)$$
$$= \big(b \wedge (a \wedge a')\big) \vee \big(a \wedge (b \wedge b')\big)$$
$$= (b \wedge 0) \vee (a \wedge 0) = 0 \vee 0$$
$$(a \wedge b) \wedge (a' \oplus b') = 0 \ldots (2)$$

$\textit{From } (1) \textit{ and } (2) \textit{ we get,}$

$$(a * b)' = a' \oplus b'$$

4. i) Let the relation R be defined on the set of all real numbers by "if x, y are real numbers, $xRy \Leftrightarrow x - y$ is a rational number". Show that R is an equivalence relation.

Solution: $\mathbb{R} - Set\ of\ all\ real\ numbers$

$$i)\forall x \in \mathbb{R}, (x - x)\ is\ also\ a\ rational\ number\ \Rightarrow (x, x) \in R$$

∴ The relation R is reflexive.

$ii)\forall x, y \in \mathbb{R}\ and\ \forall (x, y) \in R \Rightarrow (x - y) is\ arational\ number$

$$\Rightarrow (y - x)\ is\ also\ a\ rational\ number$$

$$\Rightarrow (y, x) \in R$$

∴ The relation R is symmetric.

$iii)\forall x, y, z \in \mathbb{R}, \therefore \forall (x, y), (y, z) \in R$

$\Rightarrow (x - y)\ is\ a\ rational\ number\ and\ (y - z) is\ a\ rational\ number$

$\Rightarrow (x - y) + (y - z)\ is\ also\ a\ rational\ number$

$\Rightarrow (x - z)\ a\ rational\ number$

$\Rightarrow (x, z) \in R$

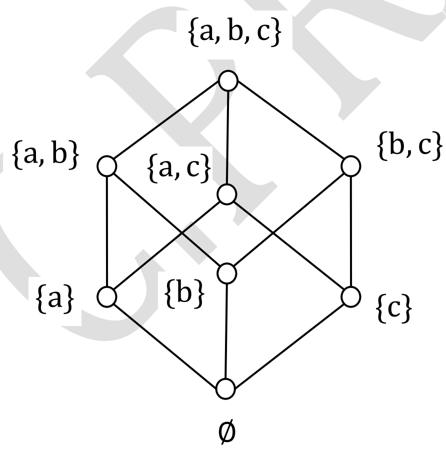
∴ The relation R is transitive.

from (i), (ii) and (iii) we get

The relation R is equivalence relation.

ii) Draw the Hasse diagram of the lattice L of all subsets of a, b, c under intersection and union.

Solution:



5. i) Define the relation P on $\{1,2,3,4\}$ by $P = \{(a, b) / |a - b| = 1\}$. Determine the adjacency matrix of P^2

Solution:

$$P = \{(1,2), (2,1), (2,3), (3,2), (3,4), (4,3)\}.$$

$$M_P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{P^2} = M_{PoP} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

ii) Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$ if $b \leq c \Rightarrow a * b \leq a * c$

$$and \;\; a \oplus b \leq a \oplus c$$

Solution:

$(a * b) * (a * c) = a * (b * a) * c = a * (a * b) * c$

$= (a * a) * (b * c) = a * b$

$$\therefore (a * b) * (a * c) = a * b$$

$$a * b \leq a * c$$

$(a \oplus b) * (a \oplus c) = a \oplus (b * c) = a \oplus c$

$$\therefore a \oplus b \leq a \oplus c$$

iii)In a distributice lattice, show that

$(a * b) \oplus (b * c) \oplus (c * a) = (a \oplus b) * (b \oplus c) * (c \oplus a)$

Solution:

$$(a * b) \oplus (b * c) \oplus (c * a) = (a * b) \oplus (c * b) \oplus (c * a)$$

$$= \big((a \oplus c) * b\big) \oplus (c * a)$$

$$= \Big(\big((a \oplus c) * b\big) \oplus c\Big) * \Big(\big((a \oplus c) * b\big) \oplus a\Big)$$

$$= \Big(\big((a \oplus c) \oplus c\big) * (b \oplus c)\Big) * \Big(\big((a \oplus c) \oplus a\big) * (b \oplus a)\Big)$$

$$= \Big(\big((a \oplus c) \oplus c\big) * (b \oplus c)\Big) * \Big(\big(a \oplus (a \oplus c)\big) * (b \oplus a)\Big)$$

$$= \Big(\big(a \oplus (c \oplus c)\big) * (b \oplus c)\Big) * \Big(\big((a \oplus a) \oplus c\big) * (b \oplus a)\Big)$$

$$= (a \oplus c) * (b \oplus c) * (a \oplus c) * (b \oplus a)$$

$$= (c \oplus a) * (b \oplus c) * (c \oplus a) * (a \oplus b)$$

$$= (b \oplus c) * (c \oplus a) * (c \oplus a) * (a \oplus b)$$

$$= (b \oplus c) * (c \oplus a) * (a \oplus b)$$

$$= (b \oplus c) * (a \oplus b) * (c \oplus a)$$

$$= (a \oplus b) * (b \oplus c) * (c \oplus a)$$

6. i) If R_1 and R_2 are equivalence relations in a set A, then prove that $R_1 \cap R_2$ is an equivalence relation in A.

Solution:

1) $\forall x \in A,\; (x, x) \in R_1 \; and \; (x, x) \in R_2 \;\; \Rightarrow (x, x) \in R_1 \cap R_2$

$$\therefore \forall x \in A, (x, x) \in R_1 \cap R_2$$

$\therefore R_1 \cap R_2$ is reflexive.

2) $\forall x, y \in A, and \;\; \forall (x, y) \in R_1 \cap R_2 \Rightarrow (x, y) \in R_1 and \; (x, y) \in R_2$

$$\Rightarrow (y, x) \in R_1 and \; (y, x) \in R_2$$

$$\Rightarrow (x, y) \in R_1 \cap R_2$$

$\therefore R_1 \cap R_2$ is symmetric.

3) $\forall x, y, z \in A, and\ \ \forall (x, y), (y, z) \in R_1 \cap R_2$

$$\Rightarrow (x, y), (y, z) \in R_1 and\ (x, y), (y, z) \in R_2$$

$$\Rightarrow (x, z) \in R_1 and\ (x, z) \in R_2$$

$$\Rightarrow (x, z) \in R_1 \cap R_2$$

$\therefore R_1 \cap R_2$ is transitive.

From (1),(2) and (3) we get

$R_1 \cap R_2$ is an equivalence relation.

ii) Simplify the Boolean expression $((x_1 + x_2) + (x_1 + x_3)) . x_1 . \overline{x_2}$

Solution:

$$\big((x_1 + x_2) + (x_1 + x_3)\big). x_1 . \overline{x_2} = (x_1 + x_2). x_1 . \overline{x_2} + (x_1 + x_3). x_1 . \overline{x_2}$$
$$= x_1 . x_1 . \overline{x_2} + x_2 . x_1 . \overline{x_2} + x_1 . x_1 . \overline{x_2} + x_3 . x_1 . \overline{x_2}$$
$$= x_1 . x_1 . \overline{x_2} + x_1 . x_2 . \overline{x_2} + x_3 . x_1 . \overline{x_2}$$
$$= x_1 . \overline{x_2} + x_1 . 0 + x_3 . x_1 . \overline{x_2}$$
$$= x_1 . \overline{x_2} + x_3 . x_1 . \overline{x_2}$$
$$= x_1 . \overline{x_2}$$

iii) State and prove the distributive inequalities of a lattice.

Solution:

Let $(L, \leq)$ be a lattice. For any $a, b, c \in L$

I) $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

II) $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

To prove $a * (b \oplus c) \geq (a * b) \oplus (a * c)$

From $a \geq a * b\ and\ a \geq a * c \Rightarrow a \geq\ (a * b) \oplus (a * c) \ldots (1)$

$$b \oplus c \geq b \geq (a * b) \ldots (2)$$
$$b \oplus c \geq c \geq (a * c) \ldots (3)$$

From (2) and (3) we get,

$$b \oplus c \geq (a * b) \oplus (a * c) \ldots (4)$$

From (1) and (4) we get,

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$

To prove $a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$

From $a \oplus b \geq a\ and\ a \oplus c \geq a\ \Rightarrow (a \oplus b) * (a \oplus c) \geq a \ldots (5)$

$$b * c \leq b \leq (a \oplus b) \ldots (6)$$
$$b * c \leq c \leq (a \oplus c) \ldots (7)$$

From (6) and (7) we get,

$$b * c \leq (a \oplus b) * (a \oplus c) \ldots (8)$$

From (5) and (8) we get,

$$a * (b \oplus c) \geq (a * b) \oplus (a * c)$$
$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)$$

7. i) If R is an equivalence relation on a set A, Prove that $[x]_R = [y]_R$ if and only if $x\ R\ y$ where $[x]_R$ and $[y]_R$ denote equivalence classes containing x and y respectively.

Proof:

Let R be an equivalence relation defined on set X.

Let $x, y, z \in X$

Let us assume that $[x]_R = [y]_R$

Let $z \in [x]_R\ then\ xRz$

$$\therefore z \in [y]_R\ then\ yRz\ (Since\ [x]_R = [y]_R)$$
$$\Rightarrow zRy\ (By\ symmetry\ of\ R)$$
$$xRz\ and\ zRy \Rightarrow xRy (By\ transitive\ of\ R)$$
$$\therefore [x]_R = [y]_R \Rightarrow xRy \ldots (1)$$

Let us assume that $xRy, so\ that\ y \in [x]_R$

Because of symmetry of $R, yRx, so\ that\ x \in [y]_R. Now\ if\ there\ is\ an\ element$

$$z \in [y]_R, then\ yRz.$$
$$xRy\ and\ yRz \Rightarrow\ xRz\ (By\ transitive\ of\ R). Thus\ z \in [x]_R$$
$$\therefore [y]_R \subseteq [x]_R \ldots (2)$$
$$By\ symmetry\ we\ also\ have\ [x]_R \subseteq [y]_R \ldots (3)$$
$$from\ (2) and\ (3) we\ get\ [x]_R = [y]_R$$
$$\therefore xRy \Rightarrow [x]_R = [y]_R \ldots (4)$$
$$from\ (1) and\ (4) we\ get$$

$[x]_R = [y]_R$ if and only if $x\ R\ y$

ii) In a lattice show that $a \le\ b\ \Leftrightarrow\ a * b =\ a\ \Leftrightarrow\ a \oplus b = b$

Solution:

To prove $a \le\ b\ \Leftrightarrow\ a * b =\ a$

Let us assume that $a \le\ b$, we know that $a \le\ a. \therefore a \le a * b \ldots (1)$

From the definition we know that $a * b \le a \ldots (2)$

From (1) and (2) we get $a * b =\ a$

$$\therefore a \le\ b\ \Rightarrow\ a * b =\ a \ldots (I)$$

Now assume that $a * b =\ a$ but it is possible iff $a \le\ b$

$$\therefore a * b =\ a\ \Rightarrow a \le\ b \ldots (II)$$

From (I) and (II) we get

$$a \le\ b\ \Leftrightarrow\ a * b =\ a$$

To prove $a * b =\ a\ \Leftrightarrow\ a \oplus b =\ b$

Let us assume that $a * b =\ a$

$$b \oplus (a * b) = b \oplus a = a \oplus b \ldots (3)$$
$$b \oplus (a * b) = b \ldots (4)$$

From (3) and (4) we get $a \oplus b =\ b$

$$\therefore a * b =\ a\ \Rightarrow\ a \oplus b =\ b \ldots (III)$$

Let us assume that $a \oplus b =\ b$

$$a * (a \oplus b) = a * b \ldots (5)$$
$$a * (a \oplus b) = a \ldots (6)$$

From (5) and (6) we get $a * b = a$

$$\therefore a \oplus b = b \Rightarrow a * b = a \ldots (IV)$$

From (III) and (IV) we get $a * b = a \Leftrightarrow a \oplus b = b$

iii) Prove that every chain is a distributive lattice.

Solution:

Let $(L, \leq)$ be a chain and $a, b, c \in L$. Consider the following cases:

(I) $a \leq b \text{ or } a \leq c$, and (II) $a \geq b \text{ and } a \geq c$

For (I)

$$a * (b \oplus c) = a \ldots (1)$$
$$(a * b) \oplus (a * c) = a \oplus a = a \ldots (2)$$

For (II)

$$a * (b \oplus c) = b \oplus c \ldots (3)$$
$$(a * b) \oplus (a * c) = b \oplus c \ldots (4)$$

∴From (1),(2) and (3),(4)

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

∴Every chain is a distributive lattice

8. i) Show that every distributive lattice is a modular. Whether the converse is true? Justify your answer

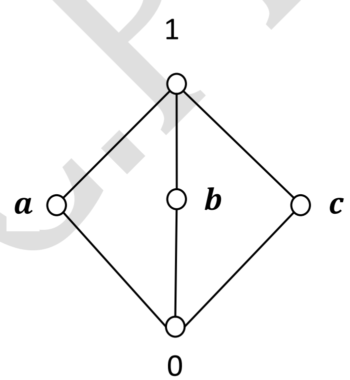
Solution:

Let $a, b, c \in L$ and assume that $a \leq c, then$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$
$$= (a \oplus b) * c$$

∴Every distributive lattice is modular.

For example let us consider the following lattice



Here in this lattice

$$\forall a, b, c \in L, a \leq b \Rightarrow a \oplus (b * c) = (a \oplus b) * c$$

∴The above lattice is modular.

$$a * (b \oplus c) = a * 1 = a \dots (1)$$
$$(a * b) \oplus (a * c) = 0 \oplus 0 = 0 \dots (2)$$

From (1) and (2) we get $a * (b \oplus c) \neq (a * b) \oplus (a * c)$

∴The above lattice is not distributive.

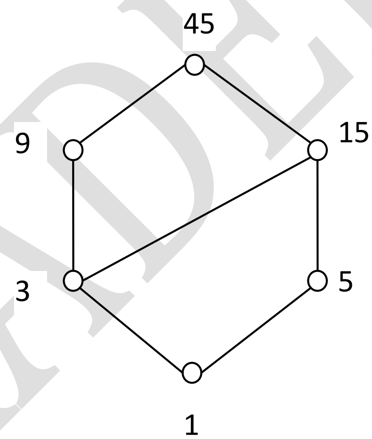
∴ Every distributive lattice is a modular but its converse is not true.

ii) Find the sub lattices of $(D_{45}, /)$. Find its complement element.

Solution:

$$D_{45} = \{1, 3, 5, 9, 15, 45\} \; under \; division \; rule$$
$$1 \oplus 45 = 45 \; and \; 1 * 45 = 1$$
$$\therefore Complement \; of \; 1 \; is \; 45$$
$$5 \oplus 9 = 45 \; and \; 5 * 9 = 1$$
$$\therefore Complement \; of \; 5 \; is \; 9$$
$$3 \oplus 15 = 15 \; and \; 3 * 15 = 3$$
$$\therefore 3 \; and \; 15 \; has \; no \; Complement$$

∴ $(D_{45}, /)$ is not a complement lattice



The sub lattices of $(D_{45}, /)$ are given below

$$S_1 = \{1, 3, 5, 9, 15, 45\},\; S_2 = \{1, 3, 9, 45\}, S_3 = \{1, 5, 15, 45\},$$
$$S_4 = \{1, 3, 5, 15\}, S_5 = \{3, 9, 15, 45\}, S_6 = \{1, 3, 9, 15, 45\},$$
$$S_7 = \{1, 3, 5, 15, 45\}, S_8 = \{1, 3\}, S_9 = \{1, 5\}, S_{10} = \{1, 3, 9\}, S_{11} = \{1, 5, 15\}$$
$$S_{12} = \{3, 9, 45\}, S_{13} = \{5, 15, 45\}, S_{14} = \{3, 9\}, S_{15} = \{5, 15\}, S_{16} = \{15, 45\}$$
$$S_{17} = \{9, 45\}, S_{18} = \{3, 15\}$$

iii) In any Boolean algebra, show that $a = b \Leftrightarrow ab' + a'b = 0$

Proof:

Case i) To prove $a = b \Rightarrow ab' + a'b = 0$

$$ab' = bb' = 0 \dots (1) [a = b \; and \; Complement \; law]$$
$$a'b = b'b = 0 \dots (2) [a = b \; and \; Complement \; law]$$
$$ab' + a'b = 0 + 0 = 0 \quad [from \; (1) \; and \; (2)]$$

Case ii) To prove $ab' + a'b = 0 \Rightarrow a = b \dots (3)$

$$ab' + a'b = 0$$

$$a + ab' + a'b = a + 0 \quad [b = c \Rightarrow a + b = a + c]$$
$$a + a'b = a \quad [Absorbtion\ lawand\ a + 0 = a]$$
$$(a + a')(a + b) = a[Distributive\ law]$$
$$1(a + b) = a \Rightarrow a + b = a \ldots (4)[Complement\ law]$$
$$Similarly\ from\ (3), we\ get\ ab' + a'b + b = 0 + b$$
$$[b = c \Rightarrow b + a = c + a]$$
$$ab' + b = b\ [Absorbtion\ law\ and\ 0 + b = b]$$
$$(a + b)(b' + b) = b[Distributive\ law]$$
$$(a + b)1 = b \Rightarrow a + b = b \ldots (5)[Complement\ law]$$

From (4) and (5) we get

$$a = b$$

9. i) Let $(L, \leq\ )$ be a lattice. For any $a, b, c \in L$ the following holds,

$$a \leq c \Leftrightarrow a \oplus (b * c) \leq (a \oplus b) * c$$

Solution: To prove $a \leq c \Rightarrow a \oplus (b * c) \leq (a \oplus b) * c$

Let us assume that $a \leq c$,

$$a \oplus (b * c) \leq (a \oplus b) * (a \oplus c)\ [Distributive\ inequality]$$
$$\leq (a \oplus b) * c\ [Distributive\ inequality]$$

To prove $a \oplus (b * c) \leq (a \oplus b) * c \Rightarrow a \leq c$

Let us assume that $a \oplus (b * c) \leq (a \oplus b) * c$

$$(a \oplus b) * (a \oplus c\ ) \leq (a \oplus b) * c[Distributive\ law]$$
$$\Rightarrow (a \oplus c\ ) \leq c \ldots (1) \left[a * b \leq a * c \Rightarrow b \leq c\right]$$
$$a \oplus (b * c) \leq (a \oplus b) * c$$
$$a \oplus (b * c) \leq (a * c) \oplus (b * c)[Distributive\ law]$$
$$\Rightarrow a \leq (a * c) \leq (a \oplus c\ ) \leq c\ [Definition\ of\ * and \oplus\ and (1)]$$
$$\Rightarrow a \leq c$$

ii) Prove that the direct product of any two distributive lattices is a distributive lattice.

Solution:

Let $(L, *, \oplus)$ $and\ (S, \wedge, \vee)\ be\ two\ lattices\ and\ let\ (L \times S, . , +)$ $be\ the\ direct\ product\ of\ two\ lattices.$

For any $(a_1, b_1), (a_2, b_2)\ and\ (a_3, b_3) \in L \times S$

$$(a_1, b_1).\left((a_2, b_2) + (a_3, b_3)\right) = (a_1, b_1).(a_2 \oplus a_3, b_2 \vee b_3)$$
$$= \left(a_1 * (a_2 \oplus a_3), b_1 \wedge (b_2 \vee b_3)\right)$$
$$= \left((a_1 * a_2) \oplus (a_1 * a_3), (b_1 \wedge b_2) \vee (b_1 \wedge b_3)\right)$$
$$= (a_1, b_1).(a_2, b_2) + (a_1, b_1).(a_3, b_3)$$

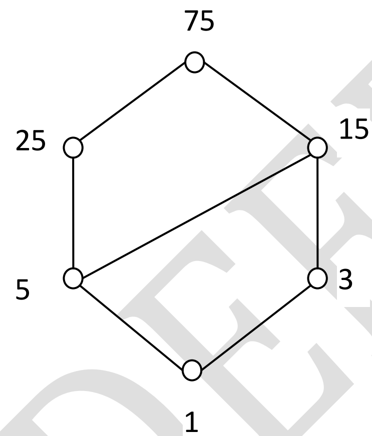
∴The direct product of any two distributive lattices is a distributive lattice.

iii) Find the complement of every element of the lattice $< S_n , D >$ for $n = 75$.

Solution:

$$S_{45} = \{1, 3, 5, 15, 25, 75\}\ under\ division\ rule$$
$$1 \oplus 75 = 75\ and\ 1 * 75 = 1$$
$$\therefore Complement\ of\ 1\ is\ 75$$
$$3 \oplus 25 = 75\ and\ 3 * 25 = 1$$
$$\therefore Complement\ of\ 3\ is\ 25$$
$$5 \oplus 15 = 15\ and\ 5 * 15 = 5$$
$$\therefore 5\ and\ 15\ has\ no\ Complement$$

∴ It is not a complement lattice



10. i) Let Z be the set of integers and let R be the relation called "congruence modulo 3" defined by

$$R = \{(x, y)\ /\ x \in Z\ \wedge\ y \in Z\ \wedge\ (x - y)\ is\ divisible\ by\ 3\}$$

   a) Prove that R is equivalence relation
   b) Determine the equivalence classes generated by the elements of Z.

Solution:

a) $i) \forall x \in Z, (x - x)\ is\ divisible\ by\ 3 \Rightarrow (x, x) \in R$

∴ The relation R is reflexive.

$ii) \forall x, y \in Z\ and\ \forall (x, y) \in R \Rightarrow (x - y) is\ divisible\ by\ 3$

$\Rightarrow (y - x) is\ also\ divisible\ by\ 3$

$\Rightarrow (y, x) \in R$

∴ The relation R is symmetric.

$iii) \forall x, y, z \in Z, \therefore \forall (x, y), (y, z) \in R$

$\Rightarrow (x - y) is\ divisible\ by\ 3\ and\ (y - z) is\ divisible\ by\ 3$

$\Rightarrow (x - y) + (y - z)\ is\ divisible\ by\ 3$

$\Rightarrow (x - z) is\ divisible\ by\ 3$

$\Rightarrow (x, z) \in R$

∴ The relation R is transitive.

from (i), (ii) and (iii) we get

The relation R is equivalence relation.

b) The equivalence classes are

$$[a]_R = \{\dots, a-2k, a-k, a, a+k, a+2k, \dots\}$$
$$where\ a = 0,1,2,\dots,k-1\ for\ congruence\ \ modulo\ k$$
$$[0]_R = \{\dots, -6, -3, 0, 3, 6, \dots\}$$
$$[1]_R = \{\dots, -5, -2, 1, 4, 7, \dots\}$$
$$[2]_R = \{\dots, -4, -1, 2, 5, 8, \dots\}$$
$$Z/R = \ \{[0]_R, [1]_R, [2]_R\}$$

ii) Write the Lattices of $(D_{35},/\ )$. Find its complements

Solution:

$$D_{35} = \{1, 5, 7, 35\}\ under\ division\ rule$$
$$1 \oplus 35 = 35\ and\ 1 * 35 = 1$$
$$\therefore Complement\ of\ 1\ is\ 35$$
$$5 \oplus 7 = 35\ and\ 5 * 7 = 1$$
$$\therefore Complement\ of\ 5\ is\ 7$$