UNIT I: RANDOM VARIABLES

PART- A -TWO MARKS

1. Given the probability density function of a continuous random variable X as follows
   \( f(x) = 6x \ (1-x) \ \text{for} \ 0 < x < 1 \). Find cumulative density function.
   
   \[
   CDF \ F(x) = \int_0^x f(x) \, dx = \int_0^x 6x \ (1-x) \, dx = 6x^2 / 2 - x^3 / 3 = 3x^2 - 2/3 \cdot x^3 , 0 < x < 1
   \]

2. A continuous random variable X can assume any value between \( x = 2 \) and \( x = 5 \) has a density function given by \( f(x) = k(x + 1) \). Find \( P(X < 4) \)

   \[
   \int_2^4 f(x) \, dx = 1 \Rightarrow k(x + 1) \, dx = k(x^2 / 2 + x) \Rightarrow k(27/2) = 1 \Rightarrow k = 2/27.
   \]

   \[
   P(X < 4) = \int_2^4 f(x) \, dx = \int_2^4 f(x+1) \, dx = 2/27 \int_2^4 (x^2 / 2 + x) = 16/27
   \]

3. If moment generating function \( M_X(t) = 1/3 \ e^{t} + 4/15 \ e^{3t} + 2/15 \ e^{4t} + 4/15 \ e^{5t} \). Find the probability mass function of X.

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<thead>
<tr>
<th>X</th>
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<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>P(X=x)</td>
<td>1/3</td>
<td>0</td>
<td>4/15</td>
<td>2/15</td>
<td>4/15</td>
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4. Suppose \( M_X(t) = (0.4 \ e^{t} + 0.6 )^8 \). Find the MGF of \( Y = 3X + 2 \)
   
   \( M_Y(t) = e^{2t} \ M_X(3t) = e^{2t} \ (0.4 \ e^{3t} + 0.6)^8 \)

5. Define MGF. Why it is called so?
   
   MFG is defined as \( M_X(t) = E(e^{tx}) \), which generates the moments of the random variable X, so it is called as Moment Generating function.

6. For a binomial distribution mean is 6 and standard deviation is \( \sqrt{2} \).
   Find the first two terms of the distribution.

   For a binomial distribution,
   
   \( \text{Mean} = np = 6 \).

   \( \text{S.D.} = \sqrt{npq} = \sqrt{2} \),

   \( npq = 2 \)

   \( 6 \times q = 2, q = 1/3 \). \( P = 1 - q = 1 - 1/3 = 2/3. \)
7. Comment of the following.“The mean of a binomial distribution is 3 and variance is 4” For a binomialDistribution, variance.
   Soln:
   np = 3, npq = 4, (3) q=
   4, q = (4/3). It is not possible.
   The given statement is wrong.

8. Find p for a binomial variate, if n=6 and 9 P(X- 4) = P(X =2)
   Soln:
   P(X=x) = 6C_x p^x q^{6-x}, 9P(X = 4) = P(X = 2),
   \[ 9 \times 6C_4 \ p^4 \ q^2 = 6C_2 \ p^2 \ q^4 \]
   \[ 9 \ p^2 = q^2, \ 9 \ p^2 (1-p)^2, \text{Since q= 1- p.} \]
   \[ 9p^2 = 1 + p^2 - 2p \]
   \[ 8p^2 + 2p -1 =0, \]
   \[ [ p+ (1/2)] [ p - (1/4)] =0, \]
   \[ P = \frac{1}{4}. \]

9. State the conditions under which the Poisson distribution is a limiting case of the Binomial Distribution.
   Soln:
   ( i ) n, the number of trials infinitely large, n→∞ (ii) p, the prob. Of success infinitely small, p→0 (iii) np =λ, a constant.

10. In a book of 520 pages, 390 typo- graphical errors occur. Assuming Poisson law for the number of errors per page, find the prob. That a random sample of 5 pages will certain no error.
11. If $X$ is the number of occurrences of the Poisson variate with mean $\lambda$, show that $P(X \geq n) - P(X \geq n+1) = P(X - n)$.

**Soln:**

$$P(X \geq n) - P(X \geq n+1) = 1 - P(X < n) - (1 - P(X < n+1))$$

$$= - P(X < n) + P(X < n+1)$$

$$= \sum_{x=0}^{n} e^{-\lambda} \frac{\lambda^x}{x!} - \sum_{x=0}^{n-1} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda} \lambda^n / n! = P(X = n).$$

12. If $X$ is Poisson variate such that $P(X=2) = \frac{2}{3} P(X=1)$ evaluate $P(X=3)$.

$$e^{-\lambda} \frac{\lambda^2}{2!} = \frac{2}{3} \{ e^{-\lambda} \lambda / 1! \}$$

$$\lambda = \frac{4}{3}.$$  

$$P(X=3) = e^{-\left(\frac{4}{3}\right)} \left(\frac{4}{3}\right)^3 / 3!$$

13. If for a Poisson variate $X$, $E(X^2) = 6$ what is $E(X)$.

$$E(X^2) = \lambda^2 + \lambda,$$  

Since Poisson distribution $\mu_2 = \lambda^2 + \lambda$

Given $E(X^2) = 6$,

$$\lambda^2 + \lambda = 6$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda = 2, -3$$

But $\lambda > 0$, $\lambda = 2$. Therefore

$$E(X) = 2.$$  

14. If $X$ is a Poisson variate with mean $\lambda$, Show that $E(X^2) = \lambda E(X+1)$. 

**Soln:**

$$\lambda = 390 / 320 = 0.75,$$  

$$P(X=x) = e^{-\lambda} \frac{\lambda^x}{x!}, x = 0, 1, 2, ...$$

$$= e^{-0.75} (0.75)^x / x!$$

Required prob. = \( P(X = 0) \)^2,

$$= (e^{-0.75})^2,$$

$$= e^{-3.75}$$
W. K. T. For a Poisson variate

\[ E(X^2) = \lambda^2 + \lambda \]

Now \( E(X+1) = E(X) + E(1) = E(X) + 1 \)
\[ = \lambda + 1 \]

Now \( E(X^2) = \lambda^2 + \lambda \)
\[ = \lambda(\lambda + 1) E(X^2) \]
\[ = \lambda E(X + 1) \]

15. Determine the distribution whose M.G.F is \( M_X(t) = e^{3(e^t - 1)} \). Also find \( P(X=1) \).

W.K.T. M.G.F of a poisson distribution

\[ M_X(t) = e^{\lambda(e^t - 1)} \]

Given \( M_X(t) = e^{3(e^t - 1)} \)
\( \lambda = 3 \)

\[ P(X=x) = e^{-\lambda} \frac{\lambda^x}{x!}, \ x = 0, 1, 2, \ldots \]

\[ P(X=x) = e^{-3} \frac{3^x}{x!}, \ x = 0, 1, 2, \ldots \]
\[ P(X=1) = e^{-3} \frac{3}{1!}, \ x = 0, 1, 2, \ldots \]
\[ = 3e^{-3} \]

16. Find the M.G.F of geometric distribution.

\[ M_X(t) = E(e^{tx}) \]
\[ = \sum_{x=0}^{\infty} e^{tx} P(X=x) \]
\[ = \sum_{x=0}^{\infty} e^{tx} \cdot p \cdot q^x \]
\[ = p \sum_{x=0}^{\infty} (e^t q)^x \]
\[ = p \left[ \frac{1 - q e^t}{1 - q e^t} \right]^{-1} \]
\[ = p / (1 - q e^t) \]

17. Identify the distribution with MGF \( M_X(t) = (5 - 4 e^t)^{-1} \)

\[ M_X(t) = \{ (1/5) / [1 - (4/5)e^t] \} \]
WKT, MGF of Geometric distribution,
$$M_X(t) = \frac{p}{1 - q \ e^t}$$

∴ The Given MGF is the MGF of geometric distribution with parameter.

$$P = 1/5, q = 4/5. $$

$$P[X=x] = (1/5) \ (4/5)^{x-1}, x = 1, 2, .......$$

18. Determine the distribution whose MGF is $M_X(t) = (1/3) \ e^t \ [e^{-t} - (2/3)]^{-1}$

Soln:
The $M_X(t)$ of negative binomial distribution with parameters $p$ and $r$ is
$$M_X(t) = p^r \ [1 - q \ e^t]^{-r},$$
Given $M_X(t) = (1/3) \ [e^t \ e^{-t} - (2/3) e^t]^{-1},$
$$= (1/3) \ [1 - (2/3) e^t]^{-1},$$
This is the M.G.F of egative Binomial Distribution ith parameters $r = 1$ and $p = 1/3.$

19. Find the MGF of a uniform distribution in $(a, b)$

$$M_X(t) = \int_a^b e^{tx} \ dx.$$ 

20. Find the MGF of a R.V. which is uniformly distributed over $(-1, 2)$

Soln:

$$M_X(t) = [1/3] = \int_{-1}^2 e^{tx} \ dx = [(e^{2t} - e^{-t}) / 3t], t \neq 0$$
$$= 1, \text{ for } t = 0$$

21. If $X$ has uniform distribution in $(-3, 3)$ find $P[X < 2]$ 

P.d.f is $f(x) = 1/6, -3 < X < 3$

$$P[X < 2] = P[0 < X < 4]$$

$$= (1/6) \ \int_0^3 dx = 1/2.$$
22. State the additive property of Gamma distribution.
   If $X_1, X_2, \ldots, X_n$ are independent Gamma variates with parameters $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ respectively, then $X_1 + X_2 + \ldots + X_n$ is also a Gamma variate with parameter $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$.

23. Mention any four properties of normal distribution.
   (1) The curve is bell shaped
   (2) Mean, median, mode coincide
   (3) All odd central moments vanish
   (4) X axis is an asymptote to the normal curve.

24. If for a normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. Find the mean and S.D. of the distribution.
   Mean = $A + \mu', 1$
   Mean = $10 + 40 = 50$.
   $\mu'$ (about the point $X = 50$) = 48.
   Since mean is 50,
   $3\sigma^2 = \mu'$
   $3\sigma^2 = 48; \sigma = 2$

25. X is a normal variate with mean 1 and variance 4, Y is another normal variate independent of X with mean 2 and variance 3, what is the distribution of $X + 2Y$?  
   $E[ X+2Y] = E[X] + 2E[Y] = 1 + 4 = 5$
   $\therefore X + 2Y$ is a normal distribution with parameter 5, 16.

26. If the continuous R.V X has p.d.f. $f(x) = 2(x+1) / 9, -1 < X < 2$. Find the p.d.f. of $Y = X^2$.
   $f_Y(y) = 2 / [9y^2], 0 < y < 1.$
   $f_Y(y) = 1 / [a(1 + (1/y^2))], 1 < y < 4.$
27. If \( X \) is uniformly distributed in (-1, 1). Find the p.d.f. of \( y = \sin \left( \frac{\pi x}{2} \right) \)

\[ f_X(x) = \frac{1}{2}, \quad -1 < x < 1. \quad dy \]

\[ \frac{dx}{dy} = \cos \left( \frac{\pi x}{2} \right) \cdot \frac{\pi}{2} \]

\[ dx / dy = 2 / \left[ \pi \sqrt{1 - y^2} \right], \quad -1 \leq y \leq 1 \]

\[ f_Y(y) = \frac{2}{\pi \sqrt{1 - y^2}}, \quad -1 < y < 1. \]

28. If \( X \) has negative binomial distribution with parameters \((n, p)\)

Prove that \( M_X(t) \) is \((Q - P e^t)^{-n}\)

\[ M_X(t) = E(e^{tx}) = \sum_{n=0}^{\infty} nC_n Q^{-n} \left( -Pe^t/Q \right) = (Q - P e^t)^{-n} \]

29. Find an expression for \( r \)th moment of weibull distribution.

\[ \mu'_r = E\left[X^r\right] = \alpha \beta \int_0^\infty x^{r+\beta-1} e^{-\alpha x^\beta} \, dx \]

Putting \( y = \alpha x^\beta \)

\[ \mu'_r = \alpha \beta \int_0^\infty y^{r+\beta-1} e^{-y} \, dy = \alpha \beta \frac{r}{\beta} \Gamma\left( \frac{r}{\beta} + 1 \right) \]

30. Mention any four properties of Normal distribution

(i) The curve is bell shaped. (ii) Mean, median, mode coincide.

(iii) All odd central moments vanish (iv) \( X \) axis is an asymptote to the normal curve.

31. Give areas of application of Binomial distribution.

The Binomial distribution finds applications in many scientific and engineering applications. Quality control measures and sampling processes in industries to classify items as defective or non-defective, medical applications as success or failure of a surgery, cure or no-cure of a patient and military applications as hit or miss of a target, are characterized by Binomial distribution.

32. Why the name negative Binomial distribution?
It is called as negative Binomial distribution because when we expand
\( p^k (1-q)^{-k} \), each term in the expansion corresponds to the value of
\( P(X=x) \) for \( x = k, k+1, k+2, \ldots \).

33. Define Geometric distribution.
If repeated independent trials can result in a success with probability \( p \) and a
failure with probability \( q = 1-p \), then probability distribution of the random
variable \( X \), the number of trials which the first success occurs is
\( P(X=x) = pq^{x-1}, x = 1, 2, 3, \ldots \).

34. Define Exponential distribution.
The continuous random variable \( X \) has an Exponential distribution if its density
function is given by
\[
 f_X(x) = \frac{1}{\beta} e^{-x/\beta}, \quad x > 0, \quad \beta > 0 \\
 = 0, \quad \text{otherwise}
\]

35. State any two properties of Normal Curve.
i. The curve is symmetric about its vertical axis through the mean \( \mu \).
ii. The normal curve approaches asymptotically the horizontal axis as \( x \)
increases in either direction away from the mean.

PART B QUESTIONS

1. A random variable \( X \) has the following probability distribution

<table>
<thead>
<tr>
<th>( X )</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X) )</td>
<td>0</td>
<td>( k )</td>
<td>2k</td>
<td>2k</td>
<td>3k</td>
<td>( k^2 )</td>
<td>2k^2</td>
<td>7k^2 + k</td>
</tr>
</tbody>
</table>

Find (i) The value of \( k \), (ii) \( P(1.5 < X < 4.5 / X > 2) \) and (iii) The smallest value of \( \lambda \) for
which \( P(X \leq \lambda) < (1/2) \).

2. Let the random variable \( X \) have the PDF \( f(x) = \lambda e^{-\lambda x}, x > 0 \). Find the moment
generating function, mean and variance.

3. Find the MGF and \( r \)th moment for the distribution whose PDF is
\( f(x) = k e^{-kx}, x > 0 \). Find also standard deviation.

4. A random variable \( X \) has the PDF \( f(x) = 2x, 0 < x < 1 \) find (i) \( P(X < \frac{1}{2}) \)
(ii) \( P(\frac{1}{4} < X < \frac{1}{2}) \) (iii) \( P(X > \frac{1}{4} / X > \frac{1}{2}) \)

5. If the density function of a continuous random variable \( X \) is given by

\[
f(x) = \begin{cases} 
ax & 0 \leq x \leq 1 \\
a & 1 \leq x \leq 2 \\
3a - ax & 2 \leq x \leq 3 \\
0 & \text{otherwise}
\end{cases}
\]
6. If the the moments of a random variable $X$ are defined by $E(X^r) = 0.6$, $r = 1,2,..$ Show that $P(X = 0) = 0.4$ $P(X = 1) = 0.6$, $P(X \geq 2) = 0$.

7. In a continuous distribution, the probability density is given by $f(x) = kx(2-x)$ $0 < x < 2$. Find $k$, mean, variance and the distribution function.

8. The cumulative distribution function of a random variable $X$ is given by

\[
F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq \frac{1}{2} \\ 1 - \frac{3}{25} \cdot (3-x)^2, & \frac{1}{2} \leq x \leq 3 \\ 1, & x \geq 3 \end{cases}
\]

Find the pdf of $X$ and evaluate $P(|X| \leq 1)$ using both the pdf and cdf.

9. Find the moment generating function of the geometric random variable with the pdf $f(x) = p \cdot q^x$, $x = 1,2,3,..$ and hence find its mean and variance.

10. Find the MGF of a binomial distribution and hence find the mean and variance.

11. Find the recurrence relation of central moments for a binomial distribution.

12. The number of monthly breakdowns of a computer is a RV having a Poisson distribution with mean equal to 1.8. Find the probability that this computer will function for a month (a) without a breakdown, (b) Wish only one breakdown, (c) Wish at least one break down.

13. Find MGF and hence find mean and variance form of binomial distribution.

14. State and prove additive property of Poisson random variable.

15. If $X$ and $Y$ are two independent Poisson random variable, then show that probability distribution of $X$ given $X+Y$ follows binomial distribution.

16. Find MGF and hence find mean and variance of a geometric distribution.

17. State and prove memory less property of a Geometric Distribution.

18. Find the mean and variance of a uniform distribution.

19. Find the MGF and hence find mean and variance of exponential distribution.

20. Find the MGF and hence find mean and variance of normal distribution.
21. Define exponential distribution and prove the memory less property.

22. In a component manufacturing industry, there is a small probability of 1/500, for any component to be defective. The components are uplided in packets f
   1. Use poisson distribution f Calculae the approximate number of packets containing (A) no defective, (B) Two defective components in a consignment of 10,000 packets.

23. The time in hours required to repair machine is exponentially distributed with parameter $\lambda = ½$. What is the probability that a repair time exceeds 2 hours? What is the conditional probability that a repair time takes at least 10 hours given that its duration exceeds 9 hours.

24. If $X$ is a binomially distributed random variable with $E(X) = 2$ and $Var. X = 4/3$. Find $P(X = 5)$