Unit-III - NUMERICAL DIFFERENTIATION

Derivatives using divided differences $(Un\ Equal\ Intervals)$

Derivatives Using Finite Differences

Newton' Forward Difference Formula To Compute The Derivatives

Consider the Newton' Forward Difference Formula

$$y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \cdots$$

$$y(x) = y_0 + \frac{p}{1!} \Delta y_0 + \frac{p^2-p}{2!} \Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!} \Delta^3 y_0 + \frac{p^4-6p^3+11p^2-6p}{4!} \Delta^4 y_0 + \cdots$$

$where\ \ p = \frac{x-x_0}{h}.$

The first derivative of $y\ at\ x = x_0 + ph\ \ is\ \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12} \Delta^4 y_0 + \cdots\right]$$

The second derivative of $y\ at\ x = x_0 + ph\ \ is$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (p-1) \Delta^3 y_0 + \frac{6p^2-18p+11}{12} \Delta^4 y_0 + \cdots\right]$$

The third derivative of $y\ at\ x = x_0 + ph\ \ is$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{12p-18}{12} \Delta^4 y_0 + \cdots\right]$$

The above formulas are used to find the derivatives nearer to $x = x_0$.

Suppose that when $\boldsymbol{x = x_0\ that\ is\ p = 0}$, the above formula become

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]\ when\ x = x_0$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 + \cdots\right]\quad when\ x = x_0$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{18}{12} \Delta^4 y_0 + \cdots\right]\qquad when\ x = x_0$$

Newton' Forward Difference Formula To Compute The Derivatives

Consider the Newton' Forward Difference Formula

$$y\,(x) = y_n + \frac{p}{1\,!}\,\nabla\, y_n + \frac{p(p+1)}{2\,!}\,\nabla^2 y_n + \frac{p(p+1)(p+2)}{3\,!}\,\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\,\nabla^4 y_n + \cdots$$

$$y\,(x) = y_n + \frac{p}{1\,!}\,\nabla\, y_n + \frac{p^2+p}{2\,!}\,\nabla^2 y_n + \frac{p^3+3p^2+2p}{3\,!}\,\nabla^3 y_n + \frac{p^4+6p^3+11p^2+6p}{4!}\,\nabla^4 y_n + \cdots$$

where $p = \frac{x-x_n}{h}$.

The first derivative of $y\ at\ x = x_n + ph\ \ is\ \frac{dy}{dx} = \frac{dy}{dp}\cdot\frac{dp}{dx}$

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla\, y_n + \frac{2p+1}{2}\,\nabla^2 y_n + \frac{3p^2+6p+2}{6}\,\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\,\nabla^4 y_n + \cdots\right]$$

The second derivative of $y\ at\ x = x_n + ph\ \ is$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\ \nabla^2 y_n + (p+1)\,\nabla^3 y_n + \frac{6p^2+18\,p+11}{12}\,\nabla^4 y_n + \cdots\right]$$

The third derivative of $y\ at\ x = x_n + ph\ \ is$

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The above formulas are used to find the derivatives nearer to $x = x_n$.

Suppose that **when** $\boldsymbol{x = x_n}$ *that is* $\boldsymbol{p = 0}$, the above formula become

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla\, y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right]\quad when\ x = x_n$$

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$$\frac{d^3y}{dx^3} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{18}{12}\,\nabla^4 y_n + \cdots\right]\qquad when\ x = x_n$$

Example : 1 Find the first and second derivative of $y\ at\, x = 15$ from the table below.

x :	15	17	19	21	23	25
y :	3.873	4.123	4.359	4.583	4.796	5.000

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
15 (x_0)	3.873(y_0)	$4.123 - 3.873 = \mathbf{0.250}$ (Δy_0)	$\mathbf{-0.014}$ $(\Delta^2 y_0)$			
				$\mathbf{0.002}$ $(\Delta^3 y_0)$		
17 (x_1)	4.123 (y_1)	$4.359 - 4.123 = 0.236$ (Δy_1)	-0.012 $(\Delta^2 y_1)$		$\mathbf{-0.001}$ $(\Delta^4 y_0)$	
				0.001 $(\Delta^3 y_1)$		$\mathbf{-0.001}(\Delta^5 y_0)$
19 (x_2)	4.359 (y_2)	$4.583 - 4.359 = 0.224$ (Δy_2)	-0.011 $(\Delta^2 y_2)$		0.001 $(\Delta^4 y_1)$	
				0.002 $(\Delta^3 y_2)$		
21 (x_3)	4.583 (y_3)	$4.796 - 4.583 = 0.213$ (Δy_3)	-0.009 $(\Delta^2 y_3)$			

23 (x_4)	4.796 (y_4)	$5.000 - 4.583 = 0.204$ (Δy_4)				
25 (x_5)	5.00 (y_5)					

To find y at $x = 15$, $that\ is$ $x = x_0$. Here $h = 2$ $(Difference\ of\ x_1 - x_0)$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \text{ when } x = x_0$$

$$\left(\frac{dy}{dx}\right)_{x=15} = \frac{1}{2}\left[0.250 - \frac{(-0.014)}{2} + \frac{0.002}{3} - \frac{(-0.001)}{4} + \frac{0.002}{5}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=15} = 0.1291c$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \text{ when } x = x_0$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=15} = -0.0046$$

Example : 2. Find the first and second derivative of y at $x = 54$ from the table below.

x :	50	51	52	53	54
y :	3.6840	3.7084	3.7325	3.7563	3.7798

Solution :

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
50 (x_0)	3.6840 (y_0)				
		$[\nabla y_1]$ 0.0244			
51 (x_1)	3.7084 (y_1)		$[\nabla^2 y_2]$ -0.0003		
		$[\nabla y_2]$ 0.0241		$[\nabla^3 y_3]$ 0.0	
52 (x_2)	3.7325 (y_2)		$[\nabla^2 y_3]$ -0.0003		$[\nabla^4 y_3]$ **0.0**
		$[\nabla y_3]$ 0.0238		$[\nabla^3 y_4]$ **0.0**	
53 (x_3)	3.7563 (y_3)		$[\nabla^2 y_4]$ **−0.0003**		
		$[\nabla y_4]$ **0.0235**			
54 (x_4)	3.7798 (y_4)				

To find y at $x = 54$, $that\ is$ $x = x_n$. Here $h = 1$ $(Difference\ of\ x_1 - x_0)$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right] \text{ when } x = x_n$$

$$\left(\frac{dy}{dx}\right)_{x=54} = \frac{1}{1}\left[0.0235 + \frac{(-0.0003)}{2} + \frac{0}{3} + 0\right]$$

$$\left(\frac{dy}{dx}\right)_{x=54} = 0.02365 - \frac{0.0003}{2}$$

$$\left(\frac{dy}{dx}\right)_{x=54} = 0.02335.$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad \text{when } x = x_n$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=54} = \frac{1}{1}\left[-0.0003 + 0\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=54} = -0.0003.$$

Example : 3. Find the value of $\sec 35^0$ using the following data.

x :	31	32	33	34
y :	0.6008	0.6249	0.6494	0.6745

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
31 (x_0)	0.6008 (y_0)			
		0.0241 (Δy_0)		
32 (x_1)	0.6249 (y_1)		**0.0004** $(\Delta^2 y_0)$	
		0.0245 (Δy_1)		**0.0002** $(\Delta^3 y_0)$
33 (x_2)	0.6494 (y_2)		0.0006 $(\Delta^2 y_1)$	
		0.0251 (Δy_2)		
34 (x_3)	0.6745 (y_3)			

To find $y \ at \ x = 31, \ that \ is \ \ x = x_0$. Here $h = 1^0 = 0.01745 \ (Difference \ of \ \ x_1 - x_0)$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \text{ when } x = x_0$$

$$\left(\frac{dy}{dx}\right)_{x=15} = \frac{1}{0.01745}\left[0.0421 - \frac{(0.0004)}{2} + \frac{0.0002}{3}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=15} = \frac{1}{0.01745}\,[\,0.023967] = 1.3732$$

$$\sec 31^0 = 1.3732$$

Example : 4. Find the first and second derivative of $y\ at\ x = 1.5\ \&\ 4.0$ from the table.

x :	1.5	2.0	5.5	3.0	3.5	4.0
y :	3.375	7.000	13.625	24.00	38.875	59.00

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5 (x_0)	$3.375 (y_0)$				
		$[\nabla y_0]$ **3.625** (Δy_0)			
2.0 (x_1)	7.000 (y_1)		$[\nabla^2 y_1]$ **3.000** $(\Delta^2 y_0)$		
		$[\nabla y_1]$ 6.625 (Δy_1)		$[\nabla^3 y_2]$ **0.75** $(\Delta^3 y_0)$	
2.5 (x_2)	13.625 (y_2)		$[\nabla^2 y_2]$ 3.750 $(\Delta^2 y_1)$		0 $(\Delta^4 y_0)$
		$[\nabla y_2]$ 10.375 (Δy_2)		$[\nabla^3 y_3]$ 0.75 $(\Delta^3 y_1)$	
3.0 (x_3)	24.00 (y_3)		$[\nabla^2 y_3]$ 4.500 $(\Delta^2 y_2)$		0 $(\Delta^4 y_1)$
		$[\nabla y_3]$ 14.875 (Δy_3)		$[\nabla^3 y_4]$ **0.75** $(\Delta^3 y_2)$	
3.5 (x_4)	38.875 (y_4)		$[\nabla^2 y_4]$ **5.250** $(\Delta^2 y_3)$		
		$[\nabla y_4]$ **20.125** (Δy_4)			
4.0 (x_5)	59.00 (y_5)				

To find $y\ at\ x = 1.5,\ that\ is\ x = x_0$. Here $h = 0.5\ (Difference\ of\ x_1 - x_0)$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right] \quad when\ x = x_0$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{(3.000)}{2} + \frac{0.75}{3}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = 4.750$$

$$And\quad \left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \cdots\right] \quad when\ x = x_0$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{0.5^2}[\,3.000 - 0.75]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = 9.000$$

To find $y \ at \ x = 4, \ that \ is \ x = x_n$. Here $h = 0.5$ $(Difference\,of\ x_1 - x_0)$

We know that

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \frac{\nabla^4 y_n}{4} + \cdots\right] \ when \ x = x_n$$

$$\left(\frac{dy}{dx}\right)_{x=4.0} = \frac{1}{0.5}\left[20.125 + \frac{5.250}{2} + \frac{0.750}{3}\right]$$

$$\left(\frac{dy}{dx}\right)_{x=4.0} = 46$$

$$And \quad \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right] \quad when \ x = x_n$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=4} = \frac{1}{0.5^2}\,[5.250 + 0.75]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=4} = 24$$

Example : 5. Find the first and second derivative of $y \ at \ x = 1.2$ from the table below.

x :	1	2	3	4	5
y :	0	1	5	6	8

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 (x_0)	$0 (y_0)$				
		1 (Δy_0)			
2 (x_1)	$1 (y_1)$		**3** $(\Delta^2 y_0)$		
		4 (Δy_1)		**−6** $(\Delta^3 y_0)$	
3 (x_2)	$5 (y_2)$		−3 $(\Delta^2 y_1)$		**10** $(\Delta^4 y_0)$
		1 (Δy_2)		4 $(\Delta^3 y_1)$	
4 (x_3)	$6 (y_3)$		1 $(\Delta^2 y_2)$		
		2 (Δy_2)			
5 (x_4)	$8 (y_4)$				

To find $y \ at \ x = 1.2,$ $[Nearer\,to\,x_0]$. Here $h = 1$ $(Difference\,of\ x_1 - x_0)$

We know that

$$\frac{dy}{dx}=\frac{1}{h}\left[\Delta y_0+\frac{2p-1}{2}\Delta^2 y_0+\frac{3p^2-6p+2}{6}\Delta^3 y_0+\frac{2p^3-9p^2+11p-3}{12}\Delta^4 y_0+\cdots\right]$$

Where $p=\frac{x-x_0}{h}$, *Since* $x=1.2,\ x_0=1,\ h=1$

$\therefore\ p=\frac{1.2-1}{1}=0.2$

$$\left(\frac{dy}{dx}\right)_{x=1.2}=\frac{1}{1}\left[1+\frac{2(0.2)-1}{2}(3)+\frac{3(0.2)^2-6(0.2)+2}{6}(-6)+\frac{2(0.2)^3-9(0.2)^2+11(0.2)-3}{12}(10)\right]$$

$$=1-0.9-0.92-0.9533$$

$$\left(\frac{dy}{dx}\right)_{x=1.2}=-1.773$$

And $$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[\Delta^2 y_0+(p-1)\Delta^3 y_0+\frac{6p^2-18p+11}{12}\Delta^4 y_0+\cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2}=\frac{1}{1}\left[3+(0.2-1)(-6)+\frac{6(0.2)^2-18(0.2)+11}{12}(-10)\right]$$

$$=3+4.8+6.366$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2}=14.17$$

Example : 6. Find the first and second derivative of y $at\ x=2.9$ from the table below.

x :	1	1.5	2	2.5	3
y :	27	106.75	324	783.75	1621

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 (x_0)	27 (y_0)				
		$[\nabla y_0]$ 79.75	$[\nabla^2 y_1]$ 137.5		
1.5 (x_1)	106.75 (y_1)			$[\nabla^3 y_2]$ 0.5	
		$[\nabla y_1]$ 217.25	$[\nabla^2 y_2]$ 242.5		**30 $(\nabla^4 y_0)$**
2 (x_2)	324 (y_2)			**$[\nabla^3 y_3]$ 135**	
		$[\nabla y_2]$ 459.75	**$[\nabla^2 y_3]$ 377.5**		
2.5 (x_3)	783.25 (y_3)				
		$[\nabla y_3]$ 837.25			
3 (x_4)	1621 (y_4)				

To find $y\ at\ x=2.9$, $[Nearer\ to\ x_n]$. Here $h=0.5$ $(Difference\ of\ x_1-x_0)$

We know that

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n + \cdots\right]$$

$Where\ \ p = \frac{x-x_n}{h},\ Since\ x = 2.9,\ x_n = 3,\ h = 0.5$

$\therefore\ p = \frac{2.9-3}{0.5} = -0.2$

$$\left(\frac{dy}{dx}\right)_{x=2.9} = \frac{1}{0.5}\left[837.5 + \frac{2(-0.2)+1}{2}(377.5) + \frac{3(-0.2)^2+6(-0.2)+2}{6}(135) + \frac{2(-0.2)^3-9(-0.2)^2+11(-0.2)+3}{12}(30)\right]$$

$$= \frac{1}{0.5}\,[\,837.5 + 113.25\ + 20.7 + 1.06]$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = 1945.02$$

$$And\qquad \frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=2.9} = \frac{1}{0.5^2}\left[377.5 + (-0.2+1)(135) + \frac{6(-0.2)^2+18(-0.2)+11}{12}(30)\right]$$

$$= \frac{1}{0.25}\,[\,377.5 + 108 + 19.1]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = 2018.4$$

Maximum and Minimum value of the given data

Example 6. Find the maximum value of y for the following data.

x :	1.2	1.3	1.4	1.5	1.6
y :	0.9320	0.9636	0.9855	0.9975	0.9996

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2 (x_0)	0.9320(y_0)	**0.0316** (Δy_0)	**−0.0097** $(\Delta^2 y_0)$	
				0 $(\Delta^3 y_0)$ *App*
1.3 (x_1)	0.9636 (y_1)	0.0219 (Δy_1)	−0.0099 $(\Delta^2 y_1)$	
				0 $(\Delta^3 y_1)$
1.4 (x_2)	0.9855 (y_2)	0.0120 (Δy_2)	−0.0099 $(\Delta^2 y_2)$	

1.5 (x_3)	0.9975 (y_3)	0.0021 (Δy_2)		
1.6 (x_4)	0.9996 (y_4)			

Let us choose $x_0 = 1.2$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\,\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\,\Delta^3 y_0 + \cdots\right]$$

For Maximum or Minimum $\frac{dy}{dx} = 0$, therefore we have

$$\Delta y_0 + \frac{2p-1}{2}\,\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\,\Delta^3 y_0 + \cdots = 0$$

$$0.0316 + \frac{2p-1}{2}(-0.0097) = 0 \;\Rightarrow\; \frac{2p-1}{2}(-0.0097) = -0.0316$$

$$\Rightarrow \frac{2p-1}{2} = -\frac{0.0316}{(-0.0097)} \;\Rightarrow\; 2p-1 = 2\,(3.2577)$$

$$\Rightarrow\; 2p = 6.5155 + 1 \;\Rightarrow\; p = \frac{7.5155}{2}$$

$$\Rightarrow\; p = 3.8\ (app)$$

$Hence\ x = x_0 + ph \;\Rightarrow\; x = 1.2 + 3.8(0.1)$

$x = 1.58$

To find Maximum value of y :

For finding the maximum value of y we use Newton's Backward interpolation formula.

We have

$$y(x) = y(x_0+ph) = y_n + p\,\nabla y_n + \frac{p\,(p+1)}{2\,!}\,\nabla^2 y_n + \cdots \quad where\ \ p = \frac{x-x_n}{h}$$

$$That\ is\ p = \frac{1.58-1.6}{0.1} = -0.2$$

$$y(x) = y(1.58) = 0.9996 + (-0.2)\,(0.0021) + \frac{(-0.2)\,(-0.2+1)}{2}\,(-0.0099)$$

$$y(x) = y(1.58) = 0.9996 - 0.0004 + 0.0008$$

$$y(x) = y(1.58) = 1$$

$\therefore$ *The Maximum value of* y *is* 1.

Example 2: Find the minimum value of y for the following data.

x :	0.60	0.65	0.70	0.75
y :	0.6221	0.6155	0.6138	0.6170

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
$0.60\ (x_0)$	$0.6221(y_0)$			
		$\mathbf{-0.0066}\ (\Delta y_0)$		
$0.65\ (x_1)$	$0.6155\ (y_1)$		$\mathbf{0.0049}\ (\Delta^2 y_0)$	
		$-0.0017\ (\Delta y_1)$		$\mathbf{0}\ (\Delta^3 y_0)$
$0.70\ (x_2)$	$0.3138\ (y_2)$		$0.0049\ (\Delta^2 y_1)$	
		$0.0032\ (\Delta y_2)$		
$0.75\ (x_3)$	$0.6170\ (y_3)$			

Let us choose $x_0 = 0.60$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots\right]$$

For Maximum or Minimum $\frac{dy}{dx} = 0$, therefore we have

$$\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \cdots = 0$$

$$-0.0066 + \left(\frac{2p-1}{2}\right)(0.0049) = 0 \implies \frac{2p-1}{2}(0.0049) = +0.0066$$

$$\Rightarrow \frac{2p-1}{2} = -\frac{0.0066}{(0.0049)} \implies 2p-1 = 2\,(1.3469)$$

$$\implies 2p = 2.6939 + 1 \implies p = \frac{3.6939}{2}$$

$$\implies p = 1.8469\ (app)$$

$Hence\ x = x_0 + ph \implies x = 1.2 + 1.8469(0.05)$

$$x = 0.6923$$

To find Maximum value of y :

For finding the minimum value of y we use **Newton's Forward interpolation** formula.

We have

$$y(x) = y(x_0 + ph) = y(x) = y_0 + p\,\nabla y_0 + \frac{p\,(p-1)}{2\,!}\nabla^2 y_0 + \frac{p\,(p-1)(p-2)}{3\,!}\nabla^3 y_0 + \cdots$$

$$That\ is\ p = \frac{0.6923 - 0.60}{0.05} = 1.8469 \qquad \left[Since \quad p = \frac{x - x_0}{h}\right]$$

$$y(x) = y(0.6923) = 0.6211 + (1.8469)(-0.0066) + \frac{(1.8469)(0.8469)}{2}(0.0049)$$

$$y(x) = y(0.6923) = 0.6211 - 0.012189 + 0.003832$$

$$y(x) = y(0.6923) = 0.6137$$

$\therefore The\ Minimum\ value\ of\ y\ is\ 0.3167.$

Derivatives at Points near the Middle of the data

Stirling's Formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots \ldots\right]$$

Example 1 : Find the First & Second derivatives of the function $y\ \ at\ \ x = 900$ from the following data.

x :	0	300	600	900	1200	1500	1800
y :	135	149	157	183	201	205	193

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0 (x_{-3})	135 (y_{-3})					
		14 (Δy_{-3})				
300 (x_{-2})	149 (y_{-2})		-6 $(\Delta^2 y_{-3})$			
		8 (Δy_{-2})		24 $(\Delta^3 y_{-3})$		
600 (x_{-1})	157 (y_{-1})		18 $(\Delta^2 y_{-2})$		-50 $(\Delta^4 y_{-3})$	
		26 (Δy_{-1})		-26 $(\Delta^3 y_{-2})$		70 $\Delta^5 y_{-3}$
900 $(\boldsymbol{x_0})$	**183** $(\boldsymbol{y_0})$		-8 $(\Delta^2 y_{-1})$		20 $(\Delta^4 y_{-2})$	
		18 $(\boldsymbol{\Delta y_0})$		-6 $(\Delta^3 y_{-1})$		-16 $\Delta^5 y_{-2}$
1200 (x_1)	201 (y_1)		**−14** $(\boldsymbol{\Delta^2 y_0})$		4 $(\Delta^4 y_{-1})$	
		4 (Δy_1)		**−2**$(\boldsymbol{\Delta^3 y_0})$		
1500 (x_2)	205 (y_2)		-16 $(\Delta^2 y_3)$			
		-12 (Δy_2)				
1800 (x_3)	193 (y_3)					

Since $x = 900$ is on the middle of the data, so we use stirling's formula

$Let\ us\ take\ \ x_0 = 900\ \ and\ \ h = 300\ \ [x_1 - x_0]$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=900} = \frac{1}{300}\left[\frac{1}{2}(18+26) - \frac{1}{12}(-6-26) + \frac{1}{60}(70-16)\right]$$

$$= \frac{1}{300}\,[22 + 2.6666 + 0.9]$$

$$\left(\frac{dy}{dx}\right)_{x=900} = 0.0852$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \cdots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=900} = \frac{1}{300}\left[-8 - \frac{1}{12}(20)\right] = -0.03222$$

Example 2 : Find the value of $f'(0.5)$ from the following data.

x :	0.35	0.40	0.45	0.50	0.55	0.60	0.65
y :	1.521	1.506	1.488	1.467	1.444	1.418	1.389

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.65 (x_{-3})	1.521 (y_{-3})					
		-0.015 (Δy_{-3})				
0.40 (x_{-2})	1.506 (y_{-2})		-0.003 $(\Delta^2 y_{-3})$			
		-0.018 (Δy_{-2})		0 $(\Delta^3 y_{-3})$		
0.45 (x_{-1})	1.488 (y_{-1})		-0.003 $(\Delta^2 y_{-2})$		0.001 $(\Delta^4 y_{-3})$	
		-0.021 (Δy_{-1})		0.001 $(\Delta^3 y_{-2})$		$-0.003\ \Delta^5 y_{-3}$
0.50 $(\boldsymbol{x_0})$	**1.467** $(\boldsymbol{y_0})$		0.002 $(\Delta^2 y_{-1})$		-0.002 $(\Delta^4 y_{-2})$	
		$\boldsymbol{-0.023}$ $(\boldsymbol{\Delta y_0})$		-0.001 $(\Delta^3 y_{-1})$		$0.003\Delta^5 y_{-2}$
0.55 (x_1)	1.444 (y_1)		$\boldsymbol{-0.003}$ $(\boldsymbol{\Delta^2 y_0})$		0.001 $(\Delta^4 y_{-1})$	
		-0.026 (Δy_1)		$\boldsymbol{0}(\boldsymbol{\Delta^3 y_0})$		
0.60 (x_2)	1.418 (y_2)		-0.003 $(\Delta^2 y_3)$			
		-0.029 (Δy_2)				
0.65 (x_3)	1.389 (y_3)					

Since $x = 1.50$ is on the middle of the data, so we use stirling's formula

$Let\ us\ take\ \ x_0 = 1.5\ \ and\ \ h = 0.05\ \ [x_1 - x_0]$

By Stirling's formula, we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots \ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.05}\left[\frac{1}{2}(-0.023 - 0.021) - \frac{1}{12}(-0.001 + 0.001) + \frac{1}{60}(0.003 - 0.003)\right]$$

$$= \frac{1}{0.05}[-0.022]$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = -0.44$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h}\left[\Delta^2 y_{-1} + \frac{1}{12}\Delta^4 y_{-2} + \cdots \ldots\right]$$

Example 3 : Find the First & Second derivatives of the function $y\ \ at\ \ x = 0.6$ from the following data.

x :	0.4	0.5	0.6	0.7	0.8
y :	1.5836	1.7974	2.0442	2.3275	2.6511

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4 (x_{-2})	1.5836 (y_{-2})				
		0.2138 (Δy_{-2})			
0.5 (x_{-1})	1.7974 (y_{-1})		0.0330 $(\Delta^2 y_{-2})$		
		0.2468 (Δy_{-1})		0.0035 $(\Delta^3 y_{-2})$	
0.6 (x_0)	2.0442 (y_0)		0.0365 $(\Delta^2 y_{-1})$		0.0003 $(\Delta^4 y_{-2})$
		0.2833 (Δy_0)		0.0038 $(\Delta^3 y_{-1})$	
0.7 (x_1)	2.3275 (y_1)		0.0403 $(\Delta^2 y_0)$		
		0.3236 (Δy_1)			
0.8 (x_2)	2.6511 (y_2)				

Since $x = 0.6$ is on the middle of the data, so we use stirling's formula

$Let\ us\ take\ \ x_0 = 0.6\ \ and\ \ h = 0.1\ \ [x_1 - x_0]$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \cdots\ldots\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.6} = \frac{1}{0.1}\left[\frac{1}{2}(0.2833 + 0.2468) - \frac{1}{12}(0.0038 + 0.0035)\right]$$

$$= \frac{1}{0.1}[0.26505 - 0.00061]$$

$$\left(\frac{dy}{dx}\right)_{x=0.6} = 2.6444$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h}\left[\Delta^2 y_{-1} + \frac{1}{12}\Delta^4 y_{-2} + \cdots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.6} = \frac{1}{0.01^2}\left[0.0365 - \frac{1}{12}(0.0003)\right]$$

$$= \frac{1}{0.01}[0.0365 - 0.000025] = \frac{1}{0.01}[0.036475]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.6} = 3.6475$$

Bessel's Formula

1. Find the value of $f'(0.04)$ using Bessel's formula, given the following data

x :	0.01	0.02	0.03	0.04	0.05	0.06
y :	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

Solution :

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$

$0.01 \ (x_{-3})$	$0.1023 \ (y_{-3})$				
		$0.0024 \ (\Delta y_{-3})$			
			$0 \ (\Delta^2 y_{-3})$		
$0.02 \ (x_{-2})$	$0.10474 \ (y_{-2})$				
		$0.0024 \ (\Delta y_{-2})$		$0.0001 \ (\Delta^3 y_{-3})$	
			$0.0001 \ (\Delta^2 y_{-2})$		$-0.0001 \ (\Delta^4 y_{-3})$
$0.03 \ (x_{-1})$	$0.1071 \ (y_{-1})$				
		$0.0025 \ (\Delta y_{-1})$		$0 \ (\Delta^3 y_{-2})$	
			$0.0001 \ (\Delta^2 y_{-1})$		$-0.0001 \ (\Delta^4 y_{-2})$
$\mathbf{0.04} \ (\boldsymbol{x_0})$	$\mathbf{0.10965} \ (\boldsymbol{y_0})$				
		$\mathbf{0.0026} \ (\boldsymbol{\Delta y_0})$		$0.0001 \ (\Delta^3 y_{-1})$	
			$\mathbf{0} \ (\boldsymbol{\Delta^2 y_0})$		
$0.05 \ (x_1)$	$0.1122 \ (y_1)$				
		$0.0026 \ (\Delta y_1)$			
$0.06 \ (x_2)$	$0.1148 \ (y_2)$				

Let us take $x_0 = 0.04$, $\quad Let \ p = \frac{x - x_0}{h}$

Since By Bessel's Formula, we have

$$\left(\frac{dy}{dx}\right) = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{3p^2 - 3p + \frac{1}{2}}{6}\ \Delta^3 y_{-1}\right]$$

$At \ x = x_0, \ p = 0$

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{4}(\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12}\Delta^3 y_{-1} + \frac{1}{24}(\Delta^3 y_{-2} + \Delta^4 y_{-1})\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.04} = \frac{1}{0.01}\left[0.0026 - \frac{1}{4}(0.0001 + 0) + \frac{1}{12}(0.0001) + \frac{1}{24}(0 + \Delta^4 y_{-1})\right]$$

$$\left(\frac{dy}{dx}\right)_{x=0.04} = 0.25625$$

TWO POINT GAUSSIAN QUADRATURE

1. Apply Gauss two point formula to evaluate $\int_{-1}^{+1} \frac{1}{1+x^2}\, dx$ **.**

Solution : Given interval is $-1 \ to \ +1$, so we apply gauss two point formula

$$\int_{-1}^{+1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right)$$

$$Here \ f(x) = \frac{1}{1+x^2}$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4} \qquad and \qquad f\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

$$\therefore \int_{-1}^{+1} \frac{1}{1+x^2}\, dx = \frac{3}{4} + \frac{3}{4} = \frac{3}{2} = 1.5$$

By actual integration :

$$\int_{-1}^{+1} \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_{-1}^{+1} = \tan^{-1}(1) - \tan^{-1}(-1)$$

$$= \tan^{-1}(1) + \tan^{-1}(1) = 2\ \tan^{-1}(1)$$

$$\int_{-1}^{1} \frac{1}{1+x^2}\, dx = 2 * \frac{\pi}{4} = \frac{\pi}{2} = 1.5708$$

The error of two point formula is $1.5708 - 1.5 = 0.0708$

2. **Apply Gauss two point formula to evaluate** $\int_0^{+1} \frac{1}{1+x^2}\, dx$ **.**

Solution :

Given interval is not $-1\ \ to\ \ +1$, so we make this as $-1\ \ to\ \ +1$

$$\int_{0}^{+1} \frac{1}{1+x^2}\, dx = \frac{1}{2} \int_{-1}^{+1} \frac{1}{1+x^2}\, dx$$

$$\int_{0}^{+1} \frac{1}{1+x^2}\, dx = \frac{1}{2}(1.5) = 0.75\ \ \textit{Since by above example.}$$

3. **Apply Gauss two point formula to evaluate** (1). $\int_{-1}^{+1} (3\,x^2 + 5\,x^4)\, dx$.

(2). $\int_0^{+1} (3\,x^2 + 5\,x^4)\, dx$.

Solution :

(1) Given interval is $-1\ \ to\ \ +1$, so we apply gauss two point formula

$$\int_{-1}^{+1} f(x)dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right)$$

$$\textit{Here}\ f(x) = 3\,x^2 + 5\,x^4$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = 3\left(\frac{-1}{\sqrt{3}}\right)^2 + 5\left(\frac{-1}{\sqrt{3}}\right)^4 = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \frac{5}{9} = \frac{14}{9} = 1.556$$

$$f\left(\frac{1}{\sqrt{3}}\right) = 3\left(\frac{1}{\sqrt{3}}\right)^2 + 5\left(\frac{1}{\sqrt{3}}\right)^4 = 3\left(\frac{1}{3}\right) + 5\left(\frac{1}{9}\right) = 1 + \frac{5}{9} = \frac{14}{9} = 1.556$$

$$\therefore \int_{-1}^{+1} (3\,x^2 + 5\,x^4)\, dx\,. = 1.556 + 1.556 = 3.112$$

(2) Given interval is not $-1\ \ to\ \ +1$, so we make it as $-1\ \ to\ \ +1$

$$\int_{0}^{+1} (3\,x^2 + 5\,x^4)\,dx = \frac{1}{2}\int_{-1}^{+1} (3\,x^2 + 5\,x^4)\,dx$$

$$\int_{0}^{+1} \frac{1}{1+x^2}\,dx = \frac{1}{2}(3.112) = 1.556 \quad \textit{Since by } (1).$$

For general range (a, b)

$$\int_{a}^{b} f(x)\,dx = \int_{-1}^{+1} f\left[\left(\frac{b-a}{2}\right) z + \left(\frac{a+b}{2}\right)\right]\left(\frac{b-a}{2}\right) dz\,.$$

$$\int_{a}^{b} f(x)\,dx = \left(\frac{b-a}{2}\right)\int_{-1}^{+1} \phi(z)\,dz\,.$$

Now $\int_{-1}^{+1} \phi(z)\,dz$ can be evaluated by using two point (or) three point Gaussian quadrature formula.

4. **Evaluate $\int_{-2}^{+2} e^{-\frac{x}{2}}\,dx$ by gauss two point formula.**

Solution :

The range is not $(-1, +1)$ so we use formula to make as $(-1, +1)$.

Here $a = -2$ & $b = +2$

$$x = \frac{b-a}{2}\,z + \frac{b+a}{2} \quad \Rightarrow \quad x = \frac{2+2}{2}\,z + \frac{2-2}{2}$$

$$x = 2\,z \;\Rightarrow\; z = \frac{x}{2} \;\Rightarrow\; dx = 2\,dz$$

$$\therefore \int_{-2}^{+2} e^{-\frac{x}{2}}\,dx = \int_{-1}^{+1} e^{-\frac{2z}{2}}\,2\,dz$$

$$\therefore \int_{-2}^{+2} e^{-\frac{x}{2}}\,dx = 2\int_{-1}^{+1} e^{-z}\,dz = 2\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right)\right]$$

$$\textit{Here } f(z) = e^{-z}$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = e^{\frac{1}{\sqrt{3}}} = 1.7813 \quad \textit{and} \quad f\left(\frac{1}{\sqrt{3}}\right) = e^{\frac{-1}{\sqrt{3}}} = 0.5614$$

$$\therefore \int_{-2}^{+2} e^{-\frac{x}{2}}\,dx = 2\int_{-1}^{+1} e^{-z}\,dz = 2\,[1.7813 + 0.5614] = 4.6854$$

$$\int_{-2}^{+2} e^{-\frac{x}{2}}\,dx = 4.6854$$

5. **Evaluate $\int_{0}^{\frac{\pi}{2}} \sin t\;dt$ by gauss two point formula.**

Solution :

The range is not $(-1, +1)$ so we use formula to make as $(-1, +1)$.

Here $a = 0$ & $b = \frac{\pi}{2}$

$$t = \frac{b-a}{2} z + \frac{b+a}{2} \quad \Rightarrow \quad t = \frac{\frac{\pi}{2} - 0}{2} z + \frac{\frac{\pi}{2} + 0}{2} \Rightarrow t = \frac{\pi}{4} z + \frac{\pi}{4}$$

$$t = \frac{\pi}{4}(z+1) \Rightarrow dt = \frac{\pi}{4}\ dz$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin t\ dt = \int_{-1}^{+1} \sin\left(\frac{\pi}{4}(z+1)\right)\left(\frac{\pi}{4}\right) dz$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin t\ dt = \frac{\pi}{4}\int_{-1}^{+1} \sin\left(\frac{\pi}{4}(z+1)\right) dz = \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right)\right]$$

$$Here\ f(z) = \sin\left(\frac{\pi}{4}(z+1)\right)$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \sin\left(\frac{\pi}{4}\left(\left[\frac{-1}{\sqrt{3}}\right] + 1\right)\right) = 0.3259$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \sin\left(\frac{\pi}{4}\left(\left[\frac{+1}{\sqrt{3}}\right] + 1\right)\right) = 0.9454$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin t\ dt = \int_{-1}^{+1} \sin\left(\frac{\pi}{4}(z+1)\right)\left(\frac{\pi}{4}\right) dz = \frac{\pi}{4}[0.3259 + 0.9454]$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin t\ dt = 0.99848$$

Actual Integral :

$$\therefore \int_0^{\frac{\pi}{2}} \sin t\ dt = [-\cos t]_0^{\frac{\pi}{2}} = -[\cos t]_0^{\frac{\pi}{2}} = -[0-1] = 1$$

$$\therefore\ The\ error\ is\ \ 0.00152$$

6. **Using Gaussian Quadrature find the value of $\int_0^{\frac{\pi}{2}} \log(1+x)\ dx$.**

Solution :

The range is not $(-1, +1)$ so we use formula to make as $(-1, +1)$.

Here $a = 0$ & $b = \frac{\pi}{2}$

$$x = \frac{b-a}{2} z + \frac{b+a}{2} \quad \Rightarrow \quad x = \frac{\frac{\pi}{2} - 0}{2} z + \frac{\frac{\pi}{2} + 0}{2} \Rightarrow x = \frac{\pi}{4} z + \frac{\pi}{4}$$

$$x = \frac{\pi}{4}(z+1) \Rightarrow dx = \frac{\pi}{4}\ dz$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(1+x)\ dx = \int_{-1}^{+1} \log\left(1+\frac{\pi}{4}\ (z+1)\right)\left(\frac{\pi}{4}\right)\ dz$$

$$\therefore \int_0^{\frac{\pi}{2}} \log(1+x)\ dx = \frac{\pi}{4}\int_{-1}^{+1} \log\left(1+\frac{\pi}{4}\ (z+1)\right)\ dz = \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right)+f\left(+\frac{1}{\sqrt{3}}\right)\right] \ldots (1)$$

$$Here\ f(z) = \log\left(1+\frac{\pi}{4}\ (z+1)\right)$$

$$f\left(\frac{-1}{\sqrt{3}}\right) = \log\left(1+\frac{\pi}{4}\left[\frac{-1}{\sqrt{3}}+1\right]\right) = 0.2866$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \log\left(1+\frac{\pi}{4}\left[\frac{+1}{\sqrt{3}}+1\right]\right) = 0.8060$$

$$\therefore\ (1) \Rightarrow \int_0^{\frac{\pi}{2}} \log(1+x)\ dx = \frac{\pi}{4}\left[f\left(-\frac{1}{\sqrt{3}}\right)+f\left(+\frac{1}{\sqrt{3}}\right)\right]$$

$$\int_0^{\frac{\pi}{2}} \log(1+x)\ dx = \frac{\pi}{4}\ [0.2866+0.8060] = 0.858$$

7. Using two – point Gaussian quadrature formula evaluate $\int_0^1 \frac{1}{1+x}\, dx$

Solution :

(i). $\int_0^1 \frac{1}{1+x}\ dx$ {Given range is not exact form}

The range is not $(-1,+1)$ so we use formula to make as $(-1,+1)$.

Here $a = 0$ & $b = 1$

$$x = \frac{b-a}{2}\ z + \frac{b+a}{2} \quad \Rightarrow \quad x = \frac{1-0}{2}\ z + \frac{1+0}{2}$$

$$x = \frac{z}{2} + \frac{1}{2} \ \Rightarrow\ \frac{z+1}{2} \ \Rightarrow\ dx = \frac{1}{2}\ dz$$

$$\therefore \int_0^1 \frac{1}{1+x}\ dx = \int_{-1}^{+1} \frac{1}{1+\left(\frac{z+1}{2}\right)}\ \left(\frac{1}{2}\right)dz = \left(\frac{1}{2}\right)\int_{-1}^{+1} \frac{1}{\left(\frac{2+z+1}{2}\right)}\ dz$$

$$= \left(\frac{1}{2}\right)\int_{-1}^{+1} \frac{1}{\left(\frac{z+3}{2}\right)}\ dz = \int_{-1}^{+1} \frac{1}{z+3}\ dz$$

$$\therefore \int_0^1 \frac{1}{1+x}\ dx = \int_{-1}^{+1} \frac{1}{z+3}\ dz$$

The integral is now in correct form $\left[\int_{-1}^{+1} \frac{1}{z+3}\ dz\right]$:

Let $f(z) = \frac{1}{z+3}$.

And $f\left(-\frac{1}{\sqrt{3}}\right) = f(-0.57735) = \frac{1}{-0.57735+3} = 0.41277$

$f\left(+\frac{1}{\sqrt{3}}\right) = f(+0.57735) = \frac{1}{+0.57735+3} = 0.27954$

$$We\ have\ \int_{-1}^{+1} f(z)dz = \left[f\left(-\frac{1}{\sqrt{3}}\right) + f\left(+\frac{1}{\sqrt{3}}\right)\right]$$

$$\therefore \int_{-1}^{+1} \frac{1}{2z+3}\ dz = 0.41277 + 0.27954 = 0.6923$$

$$\therefore \int_{0}^{1} \frac{1}{1+x}\ dx = 0.6923$$

THREE POINT GAUSSIAN QUADRATURE

$$\int_{-1}^{+1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(+\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

1. **Using Gaussian three – point formula evaluate**

(i). $\int_{-1}^{+1}[3x^2 + 5x^4]dx$ (ii). $\int_{0}^{+1}[3x^2 + 5x^4]dx$. Also compare with exact result.

Solution :

(i). $\int_{-1}^{+1}[3x^2 + 5x^4]dx$ {Given range is exact form}

Let $f(x) = 3x^2 + 5x^4$ ..

Now $f(0) = 3(0) + 5(0) = 0$

And $f\left(-\sqrt{\frac{3}{5}}\right) = 3\left(-\sqrt{\frac{3}{5}}\right)^2 + 5\left(-\sqrt{\frac{3}{5}}\right)^4 = 3\left(\frac{3}{5}\right) + 5\left(\frac{3}{5}\right)^2 = \frac{9}{5} + \frac{9}{5} = \frac{18}{5}$

$f\left(+\sqrt{\frac{3}{5}}\right) = 3\left(+\sqrt{\frac{3}{5}}\right)^2 + 5\left(+\sqrt{\frac{3}{5}}\right)^4 = 3\left(\frac{3}{5}\right) + 5\left(\frac{3}{5}\right)^2 = \frac{9}{5} + \frac{9}{5} = \frac{18}{5}$

We have $\int_{-1}^{+1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(+\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$

$$\therefore \int_{-1}^{+1}[3x^2 + 5x^4]dx = \frac{5}{9}\left[\frac{18}{5} + \frac{18}{5}\right] + \frac{8}{9}\ (0) = \frac{5}{9}\left[\frac{36}{5}\right] + 0$$

$$\therefore \int_{-1}^{+1}[3x^2 + 5x^4]dx = 4$$

Actual Integral:

$$\int_{-1}^{+1}[3x^2+5x^4]dx = 2\int_{0}^{+1}[3x^2+5x^4]dx = 2\left[3\,\frac{x^3}{3}+5\,\frac{x^5}{5}\right]_0^1$$

$$= 2[(1+1)-(0)]_0^1$$

$$\int_{-1}^{+1}[3x^2+5x^4]dx = 4 \quad \ldots(2)$$

(ii). $\int_0^{+1}[3x^2+5x^4]dx$ {Given range is not exact form}

$$\therefore \int_{0}^{+1}[3x^2+5x^4]dx = \left(\frac{1}{2}\right)\int_{-1}^{+1}[3x^2+5x^4]dx$$

$$\therefore \int_{0}^{+1}[3x^2+5x^4]dx = \left(\frac{1}{2}\right)(4) = 2 \quad \textit{Since by } (1)$$

2. Using three – point Gaussian quadrature formula evaluate

(i). $\int_{-1}^{+1}\frac{1}{1+x^2}\,dx$ **(ii).** $\int_{0}^{+1}\frac{1}{1+t^2}\,dt$ **. Also compare with exact result.**

Solution :

(i). $\int_{-1}^{+1}\frac{1}{1+x^2}\,dx$ {Given range is exact form}

Let $f(x)=\frac{1}{1+x^2}$. Now $f(0)=\frac{1}{1+0}=1$

And $f\left(-\sqrt{\frac{3}{5}}\right) = 1\Big/1+\left(-\sqrt{\frac{3}{5}}\right)^2 = 1\Big/\left(1+\frac{3}{5}\right) = 1\Big/\left(\frac{8}{5}\right) = \frac{5}{8}$

$f\left(+\sqrt{\frac{3}{5}}\right) = 1\Big/1+\left(+\sqrt{\frac{3}{5}}\right)^2 = 1\Big/\left(1+\frac{3}{5}\right) = 1\Big/\left(\frac{8}{5}\right) = \frac{5}{8}$

$$\textit{We have} \int_{-1}^{+1}f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(+\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$$

$$\therefore \int_{-1}^{+1}\frac{1}{1+x^2}dx = \frac{5}{9}\left[\frac{5}{8}+\frac{5}{8}\right]+\frac{8}{9}\;(1) = \frac{5}{9}\left[\frac{10}{8}\right]+\frac{8}{9}$$

$$\therefore \int_{-1}^{+1}\frac{1}{1+x^2}dx = \frac{50}{72}+\frac{8}{9} = 1.5833 \qquad \ldots(1)$$

Actual Integral:

$$\int_{-1}^{+1}\frac{1}{1+x^2}dx = 2\int_{0}^{+1}\frac{1}{1+x^2}dx = 2[\tan^{-1}x]_0^1$$

$$= 2[\tan^{-1} 0 - \tan^{-1} 1]_0^1 = 2\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{2}$$

$$\int_{-1}^{+1} \frac{1}{1+x^2}\,dx = 1.5708 \qquad \ldots(2)$$

(ii). $\int_0^{+1} \frac{1}{1+t^2}\,dt$ {Given range is not exact form}

$$\therefore \int_0^{+1} \frac{1}{1+t^2}\,dt = \left(\frac{1}{2}\right)\int_{-1}^{+1} \frac{1}{1+t^2}\,dt$$

$$\therefore \int_0^{+1} \frac{1}{1+t^2}\,dt = \left(\frac{1}{2}\right)[1.5833] = 0.79165$$

3. Evaluate $\int_{-1}^{+1} \frac{x^2}{1+x^4}\,dx$ by using three point Gaussian formula.

Solution :

(i). $\int_{-1}^{+1} \frac{x^2}{1+x^4}\,dx$ {Given range is exact form}

Let $f(x) = \frac{x^2}{1+x^4}$. Now $f(0) = \frac{0}{1+0} = 0$

And $f\left(-\sqrt{\frac{3}{5}}\right) = f(-0.7746) = \frac{(-0.7746)^2}{1+(-0.7746)^4} = 0.19464/_{1.3600} = 0.4412$

$f\left(+\sqrt{\frac{3}{5}}\right) = f(+0.7746) = \frac{(+0.7746)^2}{1+(+0.7746)^4} = 0.19464/_{1.3600} = 0.4412$

$$We\ have \int_{-1}^{+1} f(x)dx = \frac{5}{9}[0.4412 + 0.4412] + \frac{8}{9}(0)$$

$$\therefore \int_{-1}^{+1} \frac{x^2}{1+x^4}\,dx = \frac{5}{9}[0.88235] + 0 = 0.4902$$

4. Using three – point Gaussian quadrature formula evaluate $\int_1^5 \frac{1}{x}\,dx$

Solution :

(i). $\int_1^5 \frac{1}{x}\,dx$ {Given range is exact form}

The range is not $(-1,+1)$ so we use formula to make as $(-1,+1)$.

Here $a = 1$ & $b = 5$

$$x = \frac{b-a}{2}z + \frac{b+a}{2} \quad \Rightarrow \quad x = \frac{5-1}{2}z + \frac{5+1}{2}$$

$$x = 2z + 3 \Rightarrow dx = 2\,dz$$

$$\therefore \int_1^5 \frac{1}{x}\,dx = \int_{-1}^{+1} \frac{1}{2z+3}\,2\,dz = 2\int_{-1}^{+1} \frac{1}{2z+3}\,dz$$

$$\therefore \int_1^5 \frac{1}{x}\,dx = 2\int_{-1}^{+1} \frac{1}{2z+3}\,dz$$

The integral is now in correct form $\left[2\int_{-1}^{+1}\frac{1}{2z+3}\,dz\right]$:

Let $f(z) = \frac{1}{2z+3}$. Now $f(0) = \frac{1}{2(0)+3} = \frac{1}{3}$

And $f\left(-\sqrt{\frac{3}{5}}\right) = f(-0.7746) = \frac{1}{2(-0.7746)+3} = \frac{1}{1.4508} = 0.6893$

$f\left(+\sqrt{\frac{3}{5}}\right) = f(+0.7746) = \frac{1}{2(+0.7746)+3} = \frac{1}{4.5492} = 0.2198$

$$We\ have\ \int_{-1}^{+1} f(z)dz = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(+\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}f(0)$$

$$\therefore \int_1^5 \frac{1}{x}\,dx = 2\int_{-1}^{+1}\frac{1}{2z+3}\,dz = 2\left\{\frac{5}{9}[0.6893 + 0.2198] + \frac{8}{9}\left(\frac{1}{3}\right)\right\}$$

$$= 2\{0.505056 + 0.2963\} = 2\{0.801356\}$$

$$\therefore \int_1^5 \frac{1}{x}\,dx = 1.60272 \ \ldots(1)$$

5. **Evaluate by using three – point Gaussian quadrature formula for** $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$

Solution :

$\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ {Given range is not exact form}

The range is not $(-1,+1)$ so we use formula to make as $(-1,+1)$.

Here $a = 0$ & $b = 2$

$$x = \frac{b-a}{2}z + \frac{b+a}{2} \Rightarrow x = \frac{2-0}{2}z + \frac{2+0}{2} = 1z + 1$$

$$x = z + 1 \Rightarrow dx = dz$$

$$\therefore \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx = \int_{-1}^{+1}\frac{(z+1)^2 + 2(z+1) + 1}{1+[(z+1)+1]^4}\,dz$$

$$\int_{-1}^{+1}\frac{[z^2+2z+1]+2z+2+1}{1+[z+2]^4}\,dz = \int_{-1}^{+1}\frac{z^2+4z+4}{1+[z+2]^4}\,dz$$

$$\therefore \int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx = \int_{-1}^{+1}\frac{z^2+4z+4}{1+[z+2]^4}\,dz$$

The integral is now in correct form $\left[\int_{-1}^{+1}\frac{z^2+4z+4}{1+[z+2]^4}\,dz\right]$:

$Let\ f(z) = \frac{z^2+4z+4}{1+[z+2]^4} = \frac{(z+2)^2}{1+[z+2]^4}$

Now $f(0) = \frac{(0+2)^2}{1+[0+2]^4} = \frac{0+4}{17} = \frac{4}{17}$

And $f\left(-\sqrt{\frac{3}{5}}\right) = f(-0.7746) = \frac{([-0.7746]+2)^2}{1+[(-0.7746)+2]^4} = \frac{1.50161}{3.2548} = 0.4614$

$f\left(+\sqrt{\frac{3}{5}}\right) = f(+0.7746) = \frac{([+0.7746]+2)^2}{1+[(+0.7746)+2]^4} = \frac{7.69839}{60.2652} = 0.12774$

$$We\ have\ \int_{-1}^{+1} f(z)dz = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(+\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$$

$$\therefore \int_{-1}^{+1} f(z)dz = \frac{5}{9}\ [0.4614 + 0.12774] + \frac{8}{9}\left[\frac{4}{17}\right]$$

$$\int_{-1}^{+1} f(z)dz = \frac{5}{9}\ [0.4614 + 0.12774] + \frac{8}{9}\left(\frac{4}{17}\right) = 0.3273 + 0.2092$$

$$\therefore \int_{-1}^{+1} f(z)dz = \int_{0}^{2} \frac{x^2+2x+1}{1+(x+1)^4}\, dx = 0.5365$$

6. **Evaluate by using three – point Gaussian quadrature formula for** $\int_{0.2}^{1.5} e^{-x^2}\, dx$

Solution :

$\int_{0.2}^{1.5} e^{-x^2}\, dx$ {Given range is not exact form}

The range is not $(-1, +1)$ so we use formula to make as $(-1, +1)$.

Here $a = 0.2$ & $b = 1.5$

$$x = \frac{b-a}{2} z + \frac{b+a}{2}$$

$$x = \frac{1.5-0.2}{2} z + \frac{1.5+0.2}{2} = 0.65\, z + 0.85$$

$$x = 0.65\, z + 0.85 \Rightarrow dx = 0.65\ dz$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}\, dx = \int_{-1}^{+1} e^{-(0.65\, z+0.85)^2}\, (0.65)dz$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}\, dx = 0.65 \int_{-1}^{+1} e^{-(0.65\, z+0.85)^2}\, dz \quad ....(1)$$

The integral is now in correct form $\left[0.65 \int_{-1}^{+1} e^{-(0.65\, z+0.85)^2}\, dz\right]$:

$Let\ f(z) = e^{-(0.65\, z+0.85)^2}$.

Now $f(0) = e^{-(0.65\,[0]+0.85)^2} = e^{-(0.85)^2} = 0.4855$

And $f\left(-\sqrt{\frac{3}{5}}\right) = f(-0.7746) = e^{-(0.65\,[-0.7746]+0.85)^2} = 0.8869$

$$f\left(+\sqrt{\frac{3}{5}}\right) = f(+0.7746) = e^{-(0.65\,[+0.7746]+0.85)^2} = 0.1601$$

$$We\ have \quad \int_{-1}^{+1} f(z)dz = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(+\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9}\,f(0)$$

$$= \frac{5}{9}\,[0.8869 + 0.1601] + \frac{8}{9}\,[0.4855]$$

$$= 0.5817 + 0.4316$$

$$\int_{-1}^{+1} f(z)dz = 1.0133$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}\,dx = 0.65\int_{-1}^{+1} e^{-(0.65\,z+0.85)^2}\,dz = 0.65\,[1.0133]$$

$$\therefore \int_{0.2}^{1.5} e^{-x^2}\,dx = 0.65865$$

NUMERICAL INTEGRATION BY TRAPEZOIDAL RULE & SIMPSON'S RULE

1. Using Trapezoidal rule, evaluate $\int_{-1}^{+1} \frac{dx}{1+x^2}$ taking 8 intervals.

Solution : Here $y(x) = \frac{1}{1+x^2}$,

Since $h = Range/n$.

Range $= b - a = 1 - (-1) = 2$. So we divide the range into 8 equal intervals with $h = \frac{2}{8} = 0.25$.

We form a table

x :	-1	$-1 + 0.25 = -0.75$	$-0.75 + 0.25 = -0.5$	$-0.5 + 0.25 = -0.25$	$-0.25 + 0.25 = 0$	$0 + 0.25 = 0.25$	$0.25 + 0.25 = 0.5$	$0.5 + 0.25 = 0.75$	$+1$
y :	0.5 (y_0)	0.64 (y_1)	0.8 (y_2)	0.9412 (y_3	1 (y_4)	0.9412 (y_5	0.8 (y_6)	0.64 (y_7)	0.5 (y_8)

Trapezoidal rule

$$\int_{-1}^{+1} \frac{1}{1+x^2}\,dx = \frac{h}{2}\,[sum\ of\ first\ \&\ last\ ordinates + 2(sum\ of\ remailning\ ordinates)]$$

$$\int_{-1}^{+1} \frac{1}{1+x^2}\,dx = \frac{h}{2}\,[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)]$$

$$= \frac{0.25}{2}\,[(0.5+0.5)+2(0.64+0.8+0.9412+1+0.9412+0.8+0.64)]$$

$$= 2.3812\,[12.5248]$$

$$\int_{-1}^{+1} \frac{1}{1+x^2}\,dx = 1.5656$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h=\frac{1}{6}$ by trapezoidal rule.

Solution :

Here $y(x)=\frac{1}{1+x^2}$. Range $= b-a = 1-0 = 1$

So we divide 6 equal intervals with $h = \frac{Range}{n} = \frac{1}{6} = 0.167$.

We form a table

x :	0	0.167	0.334	0.501	0.668	0.835	+1
y :	1	0.9728	0.8996	0.7993	0.6915	0.5892	0.5

Trapezoidal rule

$$\int_{0}^{+1} \frac{1}{1+x^2}\,dx = \frac{h}{2}\,[(y_0+y_5)+2(y_1+y_2+y_3+y_4)]$$

$$\int_{0}^{+1} \frac{1}{1+x^2}\,dx = \frac{0.167}{2}\,[(1+0.5)+2(0.9728+0.8996+0.7993+0.6915+0.5892)]$$

$$= \frac{0.167}{2}\,[1.5+7.9048]$$

$$\int_{0}^{+1} \frac{1}{1+x^2}\,dx = 0.7853.$$

3. Evaluate $\int_1^2 \frac{dx}{1+x^2}$ using trapezoidal rule with two sub intervals.

Solution : Here $y(x)=\frac{1}{1+x^2}$

Range $= b-a = 2-1 = 1$

So we divide 2 equal intervals with $h=\frac{1}{2}=0.5$.

We form a table

x :	0	0.5	1
y :	1	0.8	0.5

Trapezoidal rule

$$\int_{1}^{+2} \frac{1}{1+x^2}\, dx = \frac{h}{2}\, [(y_0 + y_2) + 2(y_1)]$$

$$\int_{1}^{+2} \frac{1}{1+x^2}\, dx = \frac{\frac{1}{2}}{2}\, [(1 + 0.5) + 2(0.8)]$$

$$\int_{1}^{+2} \frac{1}{1+x^2}\, dx = 0.775.$$

4. **Dividing the range into ten equal parts, find the value of $\int_0^{\frac{\pi}{2}} \sin x\, dx$ by (1). Trapezoidal rule**

(2). Simpson's rule.

Solution : Here $y(x) = \sin x$. Range $= b - a = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

So we divide 10 equal intervals with $h = \frac{\pi}{2} \Big/ 10 = \frac{\pi}{20}$.

We form a table

x :	0	$\frac{\pi}{20}$	$\frac{2\pi}{20}$	$\frac{3\pi}{20}$	$\frac{4\pi}{20}$	$\frac{5\pi}{20}$	$\frac{6\pi}{20}$	$\frac{7\pi}{20}$	$\frac{8\pi}{20}$	$\frac{9\pi}{20}$	$\frac{10\pi}{20}$
y :	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1

Trapezoidal rule

$$\int_{0}^{\frac{\pi}{2}} \sin x\, dx = \frac{h}{2}\, [(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + Y_8 + Y_9)]\, \frac{\pi}{2} \Big/ 10$$

$$= \frac{\left(\frac{\pi}{20}\right)}{2}\, [(0 + 1) + 2(0.1564 + 0.3090 + 0.4540 + 0.5878 + 0.7071 + 0.8090 + 0.8910 + 0.9511 + 0.9877)]$$

$$\int_{0}^{\frac{\pi}{2}} \sin x\, dx = \frac{\pi}{40}\, [1 + 2(0.58531)] = \frac{\pi}{40}\, [12.7062]$$

$$\int_{0}^{\frac{\pi}{2}} \sin x\, dx = 0.9980\, .$$

(2). By Simpson's rule :

$$\int_{0}^{\frac{\pi}{2}} \sin x\, dx = \left(\frac{h}{3}\right) [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{\left(\frac{\pi}{20}\right)}{3}\, [(0 + 1) + 4(0.1564 + 0.4540 + 0.7071 + 0.8910 + 0.9877) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511)]$$

$$\int_0^{\frac{\pi}{2}} \sin x \; dx = \frac{\pi}{60} \,[1 + 4(3.1962) + 2(2.6569)]$$

$$= \frac{\pi}{60} \,[1 + 12.7848 + 5.3138] \; = \frac{\pi}{60} \,[19.0986]$$

$$\int_0^{\frac{\pi}{2}} \sin x \; dx = 1.0000$$

5. **By Simpson's one – third rule evaluate $\int_0^1 x\, e^x \, dx$ taking 4 intervals. Compare your result with actual integral.**

Solution :

Here $y(x) = x\, e^x$. Range $= b - a = 1 - 0 = 1$

So we divide 4 equal intervals with $h = \frac{1}{4} = 0.25$.

We form a table

x :	0	0.25	0.50	0.75	+1
$y = x\, e^x$:	0	0.321	0.824	1.588	2.718

Simpsons 1/3 rule

$$\int_0^{+1} x\, e^x \, dx = \frac{h}{3} \,[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$= \frac{0.25}{3} \,[(0 + 2.718) + 2(0.824) + 4(0.321 + 1.588)]$$

$$\int_0^{+1} x\, e^x \, dx = \frac{0.25}{3} \,[12.002] = \frac{3.0005}{3} = 1\,.$$

Actual integral :

$$\int_0^{+1} x\, e^x \, dx = [x\, e^x]_0^1 - \int_0^{+1} 1.e^x \, dx$$

$$= [e - 0] - [e^x]_0^1 = e - [e - 1] = 1$$

$$\int_0^{+1} x\, e^x \, dx = 1.$$

6. **Calculate $\int_{0.5}^{0.7} e^{-x}\sqrt{x}\, dx$ taking 5 ordinates by Simpson's 1/3 rule.**

Solution :

Here $y(x) = \; e^{-x}\,\sqrt{x}\,.$ Range $= b - a = 0.7 - 0.5 = 0.2$

So we divide 5 equal intervals with $h = \frac{0.2}{4} = 0.05$.

We form a table

$x:$	0.5	0.55	0.60	0.65	0.7
$y = x\,e^{x}:$	0.4289	0.4279	0.4251	0.4209	0.4155

Simpsons 1/3 rule

$$\int_{0}^{+1} f(x)dx = \frac{h}{3}\,[(y_0 + y_4) + 2(y_2) + 4(y_1 + y_3)]$$

$$\int_{0.5}^{0.7} e^{-x}\sqrt{x}\,dx = \frac{0.05}{3}\,[(0.4289 + 0.4155) + 2(0.4251) + 4(0.4279 + 0.4209)]$$

$$= \frac{0.05}{3}\,[5.0898] = \frac{0.25449}{3}$$

$$\int_{0.5}^{0.7} e^{-x}\sqrt{x}\,dx = 0.08483.$$

7. By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\ dx$ by Trapezoidal rule and Simson's rule. Verify your answer with actual integration.

Solution :

Here $y(x) = \sin x$. Range $= b - a = \pi - 0 = \pi$

So we divide 10 equal intervals with $h = \frac{\pi}{10}$.

We form a table

$x:$	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y:$	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090	0

Trapezoidal rule

$$\int_{0}^{\pi} \sin x\ dx = \frac{h}{2}\,[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + Y_8 + Y_9)]$$

$$= \frac{\left(\frac{\pi}{10}\right)}{2}\,[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1.0 + 0.9511 + 0.8090 + 0.5878 + 0.3090)]$$

$$= \frac{\pi}{20}\,[12.6276]$$

$$\int_{0}^{\pi} \sin x\ dx = 1.9843.$$

(2). By Simpson's 1/3 rule :

$$\int_{0}^{\pi} \sin x\ dx = \left(\frac{h}{3}\right)\,[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{\left(\frac{\pi}{10}\right)}{3} \left[(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)\right]$$

$$= \frac{\pi}{30} [19.0996]$$

$$\int_0^{\pi} \sin x \; dx = 2.0091.$$

By Actual Integration :

$$\int_0^{\pi} \sin x \; dx = [-\cos x]_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$

9. Evaluate $\int_0^6 \frac{1}{1+x^2}\, dx$ by (i). Trapezoidal rule (ii). Simpsons rule. Also verify by actual integration.

Solution :

Here $y(x) = \frac{1}{1+x^2}$. Range $= b - a = 6 - 0 = 6$

So we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

We form a table

x :	0	1	2	3	4	5	6
$y = \frac{1}{1+x^2}$:	1	0.500	0.200	0.100	0.058824	0.038462	0.27027

Trapezoidal rule

$$\int_0^6 \frac{1}{1+x^2}\, dx = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(1 + 0.027027) + 2(0.5 + 0.2 + 0.1 + 0.058824 + 0.038462)\right]$$

$$= \frac{1}{2}[2.821599]$$

$$\int_0^6 \frac{1}{1+x^2}\, dx = 1.4107995.$$

(2). **By Simpson's 1/3 rule** :

$$\int_0^6 \frac{1}{1+x^2}\, dx = \left(\frac{h}{3}\right)\left[(y_0 + y_6) + 2(y_1 + y_3 + y_5) + 4(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1 + 0.027027) + 2(0.2 + 0.058824) + 4(0.5 + 0.1 + 0.038462)\right]$$

$$= \frac{1}{3}[4.098523]$$

$$\int_0^{\pi} \sin x \, dx = 1.36617433.$$

(3). **By Simpson's 3/8 rule** :

$$\int_a^b f(x)\, dx = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + ..) + 2(y_3 + y_6 + ..)\right]$$

$$\int_0^6 \frac{1}{1+x^2}\, dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\left[(1 + 0.027027) + 3(0.5 + 0.2 + 0.058824 + 0.038462) + 2(0.1)\right]$$

$$\int_0^6 \frac{1}{1+x^2}\, dx = 1.357081875.$$

By Actual Integration :

$$\int_0^6 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^6 = \tan^{-1} 6 - \tan^{-1} 0 = 1.40564765$$

10. Evaluate $\int_0^5 \frac{1}{1+x^2}\, dx$ by Simpsons 1/3 rule. Also find the value of $\log_e 5$. (n=10)

Solution :

Here $y(x) = \frac{1}{1+x^2}$

Range $= b - a = 5 - 0 = 5$

So we divide 10 equal intervals with $h = \frac{5}{10} = 0.5$.

We form a table

x :	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$y = \frac{1}{1+x^2}$:	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0437	0.04

(1). By Simpson's 1/3 rule :

$$\int_0^5 \frac{1}{4x+5}\, dx = 0.4025 \quad ... (1)$$

By Actual Integration :

$$\int_0^5 \frac{1}{4x+5}\, dx = \left[\log \frac{(4x+5)}{4}\right]_0^5 = \frac{1}{4}\left[\log 25 - \log 5 - 0\right]$$

$$\int_0^5 \frac{1}{4x+5}\,dx = \frac{1}{4}\,[\log 5] \;\; ....(2)$$

From (1) & (2), we have

$$\frac{1}{4}\,[\log 5] = 0.4025 \;\; \Rightarrow \;\; \log 5 = 4(0.4025)$$

$$\log 5 = 1.61$$

Romberg's Method

1. **Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence obtain an approximation value of π.**

Solution :

Here $y(x) = \frac{1}{1+x^2}$. We take $I = \int_0^2 \frac{dx}{x^2+4}$. **Range** $= b - a = 2 - 0 = 2$

Case : I Let us take $\boldsymbol{n = 2}$. $\therefore$ $\boldsymbol{h = Range/n = \frac{2}{2} = 1}$

We form a table

$x:$	0	1	2
$y = \frac{1}{1+x^2}$	0.25	0.20	0.125

Using Trapezoidal rule :

$$I_1 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\,[(y_0 + y_2) + 2\,(y_1)]$$

$$= \frac{1}{2}\,[(0.25 + 0.125) + 2\,(0.20)]$$

$$\boldsymbol{I_1 = 0.3875.}$$

Case: II Let us take $\boldsymbol{n = 4}$. $\therefore$ $\boldsymbol{h = Range/n = \frac{2}{4} = 0.5}$.

We form a table

$x:$	0	0.5	1.0	1.5	2.0
$y = \frac{1}{1+x^2}$	0.25	0.2353	0.20	0.160	0.125

Using Trapezoidal rule :

$$I_2 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\,[(y_0 + y_4) + 2\,(y_1 + y_2 + y_3)]$$

$$= \frac{0.5}{2}\,[(0.25 + 0.125) + 2\,(0.2353 + 0.2 + 0.16)]$$

$$\boldsymbol{I_2 = 0.3914}$$

Case: III Let us take $\boldsymbol{n = 8}$. $\therefore$ $\boldsymbol{h = Range/n = 2/8 = 0.25}$.

Let us take $\boldsymbol{h = 0.25}$

We form a table

x :	0	0.25	0.50	0.75	1.0	1.25	1.50	1.75	2.0
$y = \frac{1}{1+x^2}$	0.25	0.2462	0.2353	0.2192	0.20	0.1798	0.160	0.1416	0.125

Using Trapezoidal rule :

$$I_3 = \int_0^2 \frac{dx}{x^2+4} = \frac{h}{2}\left[(y_0+y_8) + 2\,(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$= \frac{0.25}{2}\left[(0.25+0.125) + 2\,(0.2462+0.2353+0.2192+0.20+0.1798+0.16+0.1416)\right]$$

$$= (0.125)\,[3.1392]$$

$$I_3 = 0.3924$$

By using Romberg's Formula for $\boldsymbol{I_1}$ & $\boldsymbol{I_2}$ we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$I = 0.3914 + \left(\frac{0.3914 - 0.3875}{3}\right)$$

$$I = 0.3927 \qquad \ldots.(1)$$

By using Romberg's Formula for $\boldsymbol{I_2}$ & $\boldsymbol{I_3}$ we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right)$$

$$I = 0.3924 + \left(\frac{0.3924 - 0.3914}{3}\right)$$

$$I = 0.3927 \qquad \ldots.(2)$$

Since (1) & (2) are almost equal we can take

$$I = \int_0^2 \frac{dx}{x^2+4} = 0.3927 \quad \ldots.(3)$$

By Actual integral:

$$\int_0^2 \frac{dx}{x^2+4} = \int_0^2 \frac{dx}{x^2+2^2} = \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1} 0\right]$$

$$I = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{8} \qquad \ldots(4)$$

$\therefore$ *From* (3) & (4), *we get* $\frac{\pi}{8} = 0.3927 \Rightarrow \pi = 8(0.3927)$

$$\pi = 3.1416\ (App).$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Romberg's method. Hence obtain an approximation value of π.

Solution :

Here $y(x) = \frac{1}{1+x^2}$ and let $I = \int_0^1 \frac{dx}{1+x^2}$

Case : I Let us take $\boldsymbol{n} = \mathbf{2}$. $\therefore$ $\boldsymbol{h} = \boldsymbol{Range/n} = \frac{\mathbf{1}}{\mathbf{2}} = \mathbf{0.5}$

We form a table

x :	0	0.5	1
$y = \frac{1}{1+x^2}$	1	0.8	0.5

Using Trapezoidal rule :

$$I_1 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_2) + 2\,(y_1)\right]$$

$$= \frac{0.5}{2}\left[(1 + 0.5) + 2\,(1.6)\right]$$

$$I_1 = 0.775$$

Case : II Let us take $\boldsymbol{n} = \mathbf{4}$. $\therefore$ $\boldsymbol{h} = \boldsymbol{Range/n} = \frac{\mathbf{1}}{\mathbf{4}} = \mathbf{0.25}$

We form a table

x :	0	0.25	0.50	0.75	1.0
$y = \frac{1}{1+x^2}$	1	0.9412	0.80	0.64	0.5

Using Trapezoidal rule :

$$I_2 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_4) + 2\,(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}\left[(1 + 0.5) + 2\,(0.9412 + 0.80 + 0.64)\right]$$

$$I_2 = 0.7828$$

Case : III Let us take $\boldsymbol{n} = \mathbf{8}$. $\therefore$ $\boldsymbol{h} = \boldsymbol{Range/n} = \frac{\mathbf{1}}{\mathbf{8}} = \mathbf{0.125}$ We form a table

x :	0	0.125	0.25	0.375	0.50	0.625	0.750	0.875	1.0
$y = \frac{1}{1+x^2}$:	1	0.9846	0.9425	0.8767	0.80	0.7191	0.64	0.5664	0.5

Using Trapezoidal rule :

$$I_3 = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_8) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.125}{2}\left[(1 + 0.5) + 2\,(0.9846 + 0.9425 + 0.8767 + 0.80 + 0.7191 + 0.64 + 0.5664)\right]$$

$$I_3 = \int_0^1 \frac{dx}{1+x^2} = (0.125)\,[12.556]$$

$I_3 = 0.78475$

By using **Romberg's Formula fir I_1 & I_2** we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right)$$

$$I = 0.7854 \qquad \ldots.(1)$$

By using Romberg's Formula fir I_2 & I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.78475 + \left(\frac{0.78475 - 0.7828}{3}\right)$$

$$I = 0.7854 \qquad \ldots.(2)$$

Since (1) & (2) are almost equal we can take

$$I = \int_0^1 \frac{dx}{1+x^2} = 0.7854 \quad \ldots.(3)$$

By Actual integral:

$$\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = [\tan^{-1}(1) - \tan^{-1} 0]$$

$$I = \left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4} \qquad \ldots(4)$$

$\therefore$ *From* (3) & (4), *we get* $\frac{\pi}{4} = 0.7854 \Rightarrow \pi = 4(0.3927)$

$$\boldsymbol{\pi = 3.1416\ (App)}.$$

2. **Evaluate $\int_0^1 \frac{dx}{1+x}$ using Romberg's method correct to three decimal places**.

Solution :

Here $y(x) = \frac{1}{1+x}$ and let $I = \int_0^1 \frac{dx}{1+x}$

Case : I Let us take $\boldsymbol{n = 2}$. $\therefore$ $\boldsymbol{h = \frac{Range}{n} = \frac{1}{2} = 0.5}$

We form a table

x :	0	0.5	1
$y = \frac{1}{1+x}$	1	0.6666	0.5

Using Trapezoidal rule :

$$I_1 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0 + y_2) + 2\,(y_1)\right]$$

$$= \frac{0.5}{2}\left[(1 + 0.5) + 2\,(0.6666)\right]$$

$$I_1 = 0.7083$$

Case : II Let us take $\boldsymbol{n = 4}$. $\therefore$ $\boldsymbol{h = Range/n = \frac{1}{4} = 0.25}$ We form a table

x :	0	0.25	0.50	0.75	1.0
$y = \frac{1}{1+x}$	1	0.8	0.6666	0.5714	0.5

Using Trapezoidal rule :

$$I_2 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0 + y_4) + 2\,(y_1 + y_2 + y_3)\right]$$

$$= \frac{0.25}{2}\left[(1 + 0.5) + 2\,(0.8 + 0.6666 + 0.5714)\right]$$

$$I_2 = 0.6970$$

Case : III Let us take $\boldsymbol{n = 8}$. $\therefore$ $\boldsymbol{h = Range/n = \frac{1}{8} = 0.125}$ We form a table

x :	0	0.125	0.25	0.375	0.50	0.625	0.750	0.875	1.0
$y = \frac{1}{1+x^2}$:	1	0.8889	0.8	0.7273	0.6667	0.6154	0.5714	0.5333	0.5

Using Trapezoidal rule :

$$I_3 = \int_0^1 \frac{dx}{1+x} = \frac{h}{2}\left[(y_0 + y_8) + 2\,(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.125}{2}\left[(1 + 0.5) + 2\,(0.8889 + 0.8 + 0.7273 + 0.6667 + 0.6154 + 0.5714 + 0.5333)\right]$$

$$I_3 = \int_0^1 \frac{dx}{1+x} = \left(\frac{0.125}{2}\right)[11.106]$$

$$I_3 = 0.69415$$

By using Romberg's Formula fir I_1 & I_2 we have

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.6970 + \left(\frac{0.6970 - 0.7083}{3}\right)$$

$$I = 0.6932 \qquad \dots(1)$$

By using Romberg's Formula fir I_2 & I_3 we have

$$I = I_3 + \left(\frac{I_3 - I_2}{3}\right) = 0.6941 + \left(\frac{0.6941 - 0.6970}{3}\right)$$

$$I = 0.6931 \qquad \dots(2)$$

DOUBLE INTEGRALS BY USING TRAPEZOIDAL RULE & SIMPSONS RULE

1. Evaluate $\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}\,dx\,dy$ by using Trapezoidal rule & Simpson's rule.

Solution : Divide the range of x & y into 4 equal parts.

$$Let\ f(x,y)=\frac{1}{xy}\quad and\quad h=\frac{Range}{n}=\frac{b-a}{n}=\frac{2.4-2}{4}=0.1\ \ and\ k=\frac{1.4-1}{4}=0.1$$

$\therefore\ \ \boldsymbol{h=0.1}\ \ \&\ k=0.1$

	Difference $\boldsymbol{h=0.1}$					
Difference $\boldsymbol{k=0.1}$	y/x	**2 (0)**	**2.1 (1)**	**2.2 (2)**	**2.3 (3)**	**2.4 (4)**
	1.0	0.5	0.4762	0.4545	0.4348	0.4167
	1.1	0.4545	0.4329	0.4132	0.3953	0.3788
	1.2	0.4167	0.3968	0.3788	0.3623	0.3472
	1.3	0.3849	0.3663	0.3497	0.3344	0.3205
	1.4	0.3571	0.3401	0.3247	0.3106	0.2976

(i). Trapezoidal Rule:

$$Given\quad I=\int_1^{1.4}\int_2^{2.4}\frac{1}{xy}\,dx\,dy$$

$$\boldsymbol{Trapezoidal\ Rule:}\quad \boldsymbol{I=\frac{h}{2}}\,[(\boldsymbol{y_0+y_n})+\boldsymbol{2(y_1+y_2+y_3+\cdots.+y_{n-1}})]$$

First we apply Trapezoidal rule for Each **Row** $(\boldsymbol{h=0.1})$*, we have*

$$Now,\quad g_0(y)=g_0(1.0)=\frac{0.1}{2}[(0.5+0.4167)+2(0.4762+0.4545+0.4348)]\qquad [I^{st}\ \ Row]$$

$$g_0(y)=g_0(1.0)=\frac{0.1}{2}[3.6477]$$

$$g_0(y)=g_0(1.0)=0.182685$$

$$g_1(y)=g_1(1.1)=\frac{1}{0.1}[(0.4545+0.3788)+2(0.4329+0.4132+0.3953)]\qquad [2^{nd}\ \ Row]$$

$$g_1(y)=g_1(1.1)=0.165805$$

$$g_2(y)=g_2(1.2)=\frac{0.1}{2}[(0.4167+0.3472)+2(0.3968+0.3788+0.3623)]\qquad [3^{rd}\ \ Row]$$

$$g_2(y)=g_2(1.2)=0.151985$$

$$g_3(y)=g_3(1.2)=\frac{0.1}{2}[(0.3849+0.3205)+2(\ 0.3663+0.3497+0.3344)]\qquad [4^{th}\ \ Row]$$

$$g_3(y)=g_3(1.2)=0.14031$$

$$g_4(y) = g_4(1.2) = \frac{0.1}{2}[(0.3571 + 0.2976) + 2(0.3401 + 0.3247 + 0.3106)] \qquad [5^{th} \quad Row]$$

$$g_4(y) = g_4(1.2) = 0.130275$$

Applying Trapezoidal rule again for g_0, g_1, g_2, g_3 & g_4, with $k = 0.1$, we have

$$I = \int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy = \frac{h}{2}[(g_0 + g_4) + 2(g_1 + g_2 + g_3)]$$

$$I = \frac{0.1}{2}[(0.182685 + 0.130275) + 2(0.165805 + 0.151985 + 0.14031)]$$

$$\boldsymbol{I = 0.0614}$$

(ii). **Simpson's Rule**: $I = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots)]$

First we apply Simpson's rule for Each $\boldsymbol{Row}$ $(\boldsymbol{h = 0.1})$, *we have*

Now, $\quad g_0(y) = g_0(1.0) = \frac{0.1}{3}[(0.5 + 0.4167) + 4(0.4762 + 0.4348) + 2(0.4545)] \qquad [I^{st} \quad Row]$

$$= \frac{0.1}{3}[5.4697]$$

$$g_0(y) = g_0(1.0) = 0.18232$$

$$g_1(y) = g_1(1.1) = \frac{0.1}{3}[(0.4545 + 0.3788) + 4(0.4329 + 0.3953) + 2(0.4132)] \qquad [2^{nd} \quad Row]$$

$$g_1(y) = g_1(1.1) = 0.16575$$

$$g_2(y) = g_2(1.2) = \frac{0.1}{3}[(0.4167 + 0.3472) + 4(0.3968 + 0.3623) + 2(0.3788)] \qquad [3^{rd} \quad Row]$$

$$g_2(y) = g_2(1.2) = 0.15193$$

$$g_3(y) = g_3(1.2) = \frac{0.1}{3}[(0.3849 + 0.3205) + 4(0.3663 + 0.3344) + 2(0.3497)] \qquad [4^{th} \quad Row]$$

$$g_3(y) = g_3(1.2) = 0.140253$$

$$g_4(y) = g_4(1.2) = \frac{0.1}{3}[(0.3571 + 0.2976) + 4(0.3401 + 0.3106) + 2(0.3247)] \qquad [5^{th} \quad Row]$$

$$g_4(y) = g_4(1.2) = 0.13023$$

Applying Simpson's rule again for g_0, g_1, g_2, g_3 & g_4, with $k = 0.1$, we have

$$I = \int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy = \frac{h}{2}[(g_0 + g_4) + 4(g_1 + g_3) + 2(g_2)]$$

$$= \frac{0.1}{3}[(0.18232 + 0.13023) + 4(0.16575 + 0.140253) + 2(0.15193)]$$

$$\boldsymbol{I = 0.0614}$$

Actual Integration :

$$\int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy = \left(\int_1^{1.4} \frac{1}{y}\, dy\right)\left(\int_2^{2.4} \frac{1}{x}\, dx\right) = (\log y)_1^{1.4}\,(\log x)_2^{2.4}$$

$$= [\log(1.4) - \log(1)]\ [\log(2.4) - \log(2)]$$

$$= [\log(1.4) - 0]\ [\log(1.2)]$$

$$\boldsymbol{I = 0.0613}$$

2. Evaluate $\int_0^2 \int_0^2 f(x,y)\ dx\,dy$ by using Trapezoidal rule.

y/x	0 (0)	0.5 (1)	1.0 (2)	1.5 (3)	2.0 (4)
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

Solution : (i). Trapezoidal Rule: Here $\boldsymbol{h = 1}$ ***and*** $\boldsymbol{k = 0.5}$

$$Given \quad I = \int_0^2 \int_0^2 f(x,y)\ dx\,dy$$

First we apply Trapezoidal rule for Each ***Row*** $(\boldsymbol{h = 0.5})$, *we have*

$$\textbf{Trapezoidal Rule}: \quad \boldsymbol{I = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right]}$$

$$Now, \quad g_0(y) = g_0(1.0) = \frac{1}{2}[(2+5) + 2(3+4+5)] = \frac{1}{2}[31] \quad [1^{st} \quad Row]$$

$$g_0(y) = 15.5$$

$$g_1(y) = g_1(2) = \frac{1}{2}[(3+11) + 2(4+6+9)] \quad [2^{nd} \quad Row]$$

$$g_1(y) = 26$$

$$g_2(y) = g_2(3) = \frac{1}{2}[(4+14) + 2(6+8+11)] \quad [3^{rd} \quad Row]$$

$$g_2(y) = 34$$

Applying Trapezoidal rule again for $\boldsymbol{g_0, g_1, g_2}$, with $\boldsymbol{k = 1}$, we have

$$I = \int_0^2 \int_0^2 \boldsymbol{f(x,y)\,dx\,dy} = \frac{h}{2}\left[(g_0 + g_2) + 2(g_1)\right]$$

$$I = \frac{1}{2}\left[(15.5 + 34) + 2(26)\right]$$

$$\boldsymbol{I = 20.75}$$

3. Using Simpson's 1/3 rule, evaluate $\int_0^1 \int_0^1 \frac{1}{1+x+y}\ dx\,dy$ taking $h = k = 0.5$.

Solution : Divide the range of x & y into 4 equal parts.

$h = k = 0.5$ Let $f(x,y) = \frac{1}{1+x+y}$.

y/x	0 (0)	0.5 (1)	1.0 (2)
0	**1**	**0.6667**	**0.5**
0.5	**0.6667**	**0.5**	**0.4**

1.0	**0.5**	**0.4**	**0.3333**

Simpson's Rule:

$$Given \quad I = \int_0^1 \int_0^1 \frac{1}{1+x+y}\, dx\, dy$$

First we apply Simpson's rule for Each $\boldsymbol{Row}$ $(\boldsymbol{h = 0.5})$*, we have*

$$Simson's\ Rule: \quad I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots)\right]$$

$$Now, \quad g_0(y) = g_0(0) = \frac{0.5}{3}[(1 + 0.5) + 4(0.6667) + 2(0)] \qquad [I^{st} \quad Row]$$

$$g_0(y) = 0.694466$$

$$g_1(y) = g_1(0.5) = \frac{0.5}{3}[(0.6667 + 0.4) + 4(0.5) + 2(0)] \qquad [2^{nd} \quad Row]$$

$$g_1(y) = 0.51112$$

$$g_2(y) = g_2(1.0) = \frac{0.5}{3}[(0.5 + 0.3333) + 4(0.4) + 2(0)] \qquad [3^{rd} \quad Row]$$

$$g_2(y) = 0.40555$$

Applying Simpson's rule again for $\boldsymbol{g_0, g_1, g_2}$, with $\boldsymbol{k = 0.5}$, we have

$$I = \int_0^1 \int_0^1 \frac{1}{1+x+y}\, dx\, dy = \frac{h}{2}\left[(g_0 + g_4) + 4(g_1 + g_3) + 2(g_2)\right]$$

$$I = \frac{0.5}{3}[(0.694466 + 0.40555) + 4(0.51112) + 2(0)]$$

$$I = 0.52408$$

4. Evaluate $\int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\, dx\, dy$ **by using Trapezoidal rule with** $\boldsymbol{h = k = 0.25}$**.**

Solution :

Let $f(x,y) = \frac{2xy}{(1+x^2)(1+y^2)}$ and $h = k = 0.25$

	Difference $h = 0.25$					
Difference $k = 0.25$	y / x	1 (0)	1.25 (1)	1.50 (2)	1.75 (3)	2.0 (4)
	0	0	0	0	0	0
	0.25	0.2353	0.2295	0.2172	0.2027	0.1882
	0.50	0.4	0.3902	0.3692	0.3446	0.32
	0.75	0.48	0.5854	0.4431	0.4135	0.384
	1	0.5	0.4878	0.4615	0.4308	0.4

Trapezoidal Rule:

$$Given \quad I = \int_0^1 \int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\, dx\, dy$$

First we apply Trapezoidal rule for Each $\boldsymbol{Row}$ $(\boldsymbol{h = 0.25})$*, we have*

$Trapezoidal\ Rule: \quad I = \frac{h}{2}\,[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots . + y_{n-1})]$

$Now, \quad g_0(y) = g_0(0) = \frac{0.25}{2}[(0+0) + 2(0)] \qquad [I^{st} \quad Row]$

$g_0(y) = 0$

$g_1(y) = g_1(0.25) = \frac{0.25}{2}[(0.2353 + 0.1882) + 2(0.2295 + 0.2172 + 0.2027)] \qquad [2^{nd} \quad Row]$

$g_1(y) = 0.2152875$

$g_2(y) = g_2(0.50) = \frac{0.25}{2}[(0.4 + 0.32) + 2(\,0.3902 + 0.3692 + 0.3446)] \qquad [3^{rd} \quad Row]$

$g_2(y) = 0.3660$

$g_3(y) = g_3(0.75) = \frac{0.25}{2}[(0.48 + 0.384) + 2(\,0.468 + 0.4431 + 0.4135)] \qquad [4^{th} \quad Row]$

$g_3(y) = 0.439$

$g_4(y) = g_4(1) = \frac{0.25}{2}[(0.5 + 0.4) + 2(0.4878 + 0.4615 + 0.4308\,)] \qquad [5^{th} \quad Row]$

$g_4(y) = 0.457525$

Applying Trapezoidal rule again for g_0, g_1, g_2, g_3 & g_4, with $k = 0.25$, we have

$$I = \int_0^1 \int_1^2 \frac{2\,xy}{(1+x^2)(1+y^2)}\,dx\,dy = \frac{h}{2}\,[(g_0 + g_4) + 2(g_1 + g_2 + g_3\,)]$$

$$I = \frac{0.25}{2}\,[(0 + 0.457525) + 2(0.2152875 + 0.3660 + 0.4685\,)]$$

$$\boldsymbol{I = 0.3122}$$

5. Evaluate $\int_1^2 \int_1^2 \frac{1}{x^2+y^2}\,dx\,dy$ by taking $h = 0.2$ along $x - direction\ and\ k = 0.25\ along\ y - direction.$

Solution : Let $f(x,y) = \frac{1}{x^2+y^2}$ and $h = 0.2\ \ and\ \ k = 0.25$

y / x	1 (0)	1.2 (1)	1.4 (2)	1.6 (3)	1.8 (4)	2.0 (5)
1	0.5	0.4098	0.3378	0.2809	0.2359	0.2
1.25	0.3902	0.3331	0.2839	0.2426	0.2082	0.1798
1.50	0.3077	0.2710	0.2375	0.2079	0.1821	0.16
1.75	0.2462	0.2221	0.1991	0.1779	0.1587	0.1416
2	0.2	0.1838	0.1679	0.1524	0.1381	0.125

(i). **Trapezoidal Rule:** $I = \int_1^2 \int_1^2 \frac{1}{x^2+y^2}\,dx\,dy$

First we apply Trapezoidal Rule for Each **Row** $(h = 0.25)$, *we have*

$Trapezoidal\ Rule: \quad I = \frac{h}{2}\,[(y_0 + y_n) + (y_1 + y_2 + y_3 + \cdots)]$

$Now, \quad g_0(y) = g_0(1.0) = \frac{0.25}{2}[(0.5 + 0.2) + (0.4098 + 0.3378 + 0.2809 + 0.2359)] \quad [I^{st} \quad Row]$

$g_0(y) = 0.40918$

$g_1(y) = g_1(1.25) = \frac{0.25}{2}[(0.3902 + 0.1798) + (0.3331 + 0.2839 + 0.2426 + 0.2082)] \quad [2^{nd} \quad Row]$

$g_1(y) = 0.32142$

$g_2(y) = g_2(1.50) = \frac{0.25}{2}[(0.3077 + 0.16) + (0.2710 + 0.2839 + 0.2079 + 0.1821)] \quad [3^{rd} \quad Row]$

$g_2(y) = 0.26854$

$g_3(y) = g_3(1.75) = \frac{0.25}{2}[(0.2462 + 0.1416) + (0.2221 + 0.1991 + 0.1779 + 0.1587)] \quad [4^{th} \quad Row]$

$g_3(y) = 0.2253$

$g_4(y) = g_4(2.0) = \frac{0.25}{3}[(0.2 + 0.125) + (0.1838 + 0.1679 + 0.1524 + 0.1381)] \quad [5^{th} \quad Row]$

$g_4(y) = 0.19015$

Applying Trapezoidal rule again for $\boldsymbol{g_0, g_1, g_2, g_3}$ & $\boldsymbol{g_4}$, with $k = \mathbf{0.25}$, we have

$$I = \int_1^{1.4}\int_2^{2.4} \frac{1}{xy}\, dx\, dy = \frac{k}{2}\,[(g_0 + g_4) + 4(g_1 + g_3) + 2(g_2)]$$

$$I = \frac{0.25}{2}[(0.40918 + 0.19015) + (0.32142 + 0.2253 + 0.26854)]$$

$$\boldsymbol{I = 0.2769}$$

6. Using Simpson's rule, evaluate $\int_1^2\int_1^2 \frac{1}{x+y}\,dx\,dy$ by dividing the interval $(1,2)$ into two sub intervals.

Solution :

Let $f(x,y) = \frac{1}{x+y}$ and $h = k = \frac{2-1}{2} = 0.5$

$y\,/\,x$	1	1.5	2
1	0.5	0.4	0.3333
1.5	0.4	0.3333	0.2857
2	0.3333	0.2857	0.25

Simpson's Rule:

$$I = \int_1^2\int_1^2 \frac{1}{x+y}\, dx\, dy$$

First we apply Simson's Rule for Each $\boldsymbol{Row}$ $(\boldsymbol{h = 0.5})$, *we have*

$$Simson's\ Rule: \quad I = \frac{h}{3}\,[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots)]$$

$Now, \quad g_0(y) = g_0(1.0) = \frac{0.5}{3}[(0.5 + 0.3333) + 4(0.4) + 2(0)] \quad [I^{st} \quad Row]$

$g_0(y) = 0.40555$

$$g_1(y) = g_1(2) = \frac{0.5}{3}[(0.4 + 0.2857) + 4(0.3333) + 2(0)] \qquad [2^{nd} \quad Row]$$

$$g_1(y) = 0.33468$$

$$g_2(y) = g_2(3) = \frac{0.5}{3}[(0.3333 + 0.25) + 4(0.2857) + 2(0)] \qquad [3^{rd} \quad Row]$$

$$g_2(y) = 0.28768$$

Applying Simpson's rule again for g_0, g_1, g_2, with $k = 0.25$, we have

$$I = \int_1^2 \int_1^2 \frac{1}{x+y}\, dx\, dy = \frac{k}{3}[(g_0 + g_2) + 4(g_1) + 2(0)]$$

$$I = \frac{0.5}{3}[(0.40555 + 0.28768) + 4(0.33468) + 2(0)]$$

$$I = 0.33865$$

7. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sqrt{\sin(x+y)}\, dx\, dy$ by numerical double integration using Simpson's rule with $h = k = \frac{\pi}{4}$

Solution : Let $f(x, y) = \sqrt{\sin(x+y)}$ and $h = k = \frac{\pi}{4}$

y / x	0	$\pi/4$	$\pi/2$
0	0	0.8409	1
$\pi/4$	0.8409	1	0.8409
$\pi/2$	1	0.8409	0

Simpson's Rule:

$$I = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sqrt{\sin(x+y)}\, dx\, dy$$

First we apply Simson's Rule for Each **Row** $(h = \pi/4)$, *we have*

$$Simson's\ Rule: \quad I = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + \cdots)]$$

$$Now, g_0(y) = g_0(0) = \frac{\left(\frac{\pi}{4}\right)}{3}[(0 + 1) + 4(0.8409) + 2(0)] \qquad [I^{st} \quad Row]$$

$$g_0(y) = 1.142388$$

$$g_1(y) = g_1\left(\frac{\pi}{4}\right) = \frac{\left(\frac{\pi}{4}\right)}{3}[(0.8409 + 0.8409) + 4(1) + 2(0)] \qquad [2^{nd} \quad Row]$$

$$g_1(y) = 1.48749$$

$$g_2(y) = g_2\left(\frac{\pi}{2}\right) = \frac{\left(\frac{\pi}{4}\right)}{3}[(1 + 0) + 4(0.8409) + 2(0)] \qquad [3^{rd} \quad Row]$$

$$g_2(y) = 1.142388$$

Applying Simpson's rule again for g_0, g_1, g_2, with $k = \pi/4$, we have

$$I=\int_1^{\frac{\pi}{2}}\int_1^{\frac{\pi}{2}}\sqrt{\sin(x+y)}\,dx\,dy=\frac{h}{3}\left[(g_0+g_2)+4(g_1)+2(0)\right]$$

$$I=\frac{\left(\frac{\pi}{4}\right)}{3}\left[(1.142388+1.142388)+4(1.4874)+2(0)\right]$$

$$\boldsymbol{I=2.15575}$$