Anna University
April/May 2010
Part-A

1. Obtain the mean for a geometric random variable.

Solution:

Let X be a geometric random variable.

The probability mass function is given by

$$P(X = r) = q^{r-1}p, r = 1,2, \dots$$

$$E(x) = \sum_{x=-\infty}^{\infty} xP(X = x) = \sum_{x=1}^{\infty} xq^{x-1}p = p\sum_{x=1}^{\infty} xq^{x-1}$$

$$= p(1 + 2q + 3q^2 + 4q^3 + \cdots) = p(1 - q)^{-2} = pp^{-2}$$

$$E(x) = \frac{1}{p}$$

2. What is meant by memoryless property? Which continuous distribution follows this property?

Solution:

If X is a random variable then for any two positive integers m and n

$P(X > m + n/X > m) = P(X > n)$ which is the memoryless property.

Exponential distribution follows this property.

3. Give a real life example each for positive correlation and negative correlation.

Solution:

If years of experience increases, then the salary of employees increases in the company is an example for positive correlation.

If the availability increases then the demand will decrease is an example for negative correlation.

4. State central limit theorem for independent and identically distributed random variables.

Solution:

If $X_1, X_2, X_3, \dots, X_n$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu$ and $var(X_i) = \sigma^2, i = 1,2, \dots, n$ and if $S_n = X_1 + X_2 + \cdots + X_n$ then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to infinity.

5. Is a Poisson process a continuous time Markov chain? Justify your answer.

Ans:

The Poisson process is a Markov process.

$$P[X(t_3) = n_3/X(t_2) = n_2, X(t_1) = n_1] = \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_2) = n_2, X(t_1) = n_1]}$$

$$= \frac{e^{-\lambda(t_3-t_2)}\lambda^{(n_3-n_2)}(t_3 - t_2)^{(n_3-n_2)}}{(n_3 - n_2)!}$$

$$P[X(t_3) = n_3/X(t_2) = n_2, X(t_1) = n_1] = P[X(t_3) = n_3/X(t_2) = n_2]$$

Therefore the conditional probability depends only on the most recent value.
Therefore the Poisson process is a continuous time Markov chain.

6. Consider the Markov chain consisting of the three states $\mathbf{0, 1, 2}$ and transition probability matrix

$$\boldsymbol{P} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{vmatrix}$$

is reducible? Justify.

Solution:

$$P^2 = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{vmatrix} \begin{vmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{3}{8} & \frac{1}{8} \\ \frac{3}{8} & \frac{19}{48} & \frac{11}{48} \\ \frac{1}{6} & \frac{11}{36} & \frac{19}{36} \end{vmatrix}$$

$$P_{00}^{(1)} > 0, P_{01}^{(1)} > 0, P_{02}^{(2)} > 0, P_{10}^{(1)} > 0, P_{11}^{(1)} > 0, P_{12}^{(1)} > 0, P_{20}^{(2)} > 0, P_{21}^{(1)} > 0, P_{22}^{(1)} > 0$$

Hence Markov chain is irreducible.

7. Suppose that customers arrive at a Poisson rate of one per every 12 minutes and that the service time is exponential at a rate of one service per 8 minutes. What is the average number of customers in the system?

Solution:

$$\lambda = \frac{1}{12}\, customer/min$$

$$\mu = \frac{1}{8}\, customer/min$$

The average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{12}}{\frac{1}{8} - \frac{1}{12}} = \frac{8}{4} = 2\ customers\ per\ minute.$$

8. Define $\boldsymbol{M/M/2}$ queueing model. Why the notation $\boldsymbol{M}$ is used?

Solution:

$M/M/2$ is a two server Poisson queue model, where the arrival rate follows Poisson distribution and the service rate follows exponential distribution. The notation M stands for Markov.

9. Distinguish between open and closed networks.

Ans:

Open Networks	Closed Networks
Arrival from outside the node i is allowed	New customers never enter in to the system
Once the customer gets the service completed at node i, he joins the queue at node j with probability P_{ij} or leaves the system with probability P_{i0}.	Existing customer never depart from the system $P_{i0} = 0$ and $r_i = 0$ for all i.

10. $M/G/1$ queueing system is Markovian. Comment on this statement.

Ans:

$M/G/1$ queueing system is a non-Markovian queueing model. Here G indicates that the service time follows a general distribution.

Part-B

11. (a) (i) By calculating the moment generating function of Poisson distribution with parameter λ, prove that the mean and variance of the Poisson distribution are same.

Solution:

The probability function of Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,\dots, \quad \lambda > 0.$$

Moment generating function is given by

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$M_X(t) = e^{-\lambda}\left(e^{e^t\lambda}\right) = e^{\lambda(e^t-1)} \qquad \left[Since \sum_{n=0}^{\infty} \frac{(x)^n}{n!} = e^x\right]$$

$$M_X'(t) = e^{\lambda(e^t-1)} e^t \lambda$$

$$M_X'(t) = e^{\lambda(e^t-1)}(e^t\lambda)^2 + e^{\lambda(e^t-1)} e^t \lambda$$

$$E(x) = M_X'(0) = e^{\lambda(e^0-1)} e^0 \lambda = \lambda$$

$$E(x^2) = M_X''(0) = e^{\lambda(e^0-1)} e^0 \lambda + e^{\lambda(e^0-1)}(e^0\lambda)^2 = \lambda + \lambda^2$$

$$var(x) = E(x^2) - (E(x))^2 = \lambda + \lambda^2 - \lambda^2 = \lambda$$

$$Mean = variance = \lambda$$

11.(a)(ii) If the density function of X equals

$$f(x) = \begin{cases} ce^{-2x}, & 0 < x < \infty \\ 0, & Otherwise \end{cases}$$

Find C. What is $P(X > 2)$?

Solution:

Given $f(x)$ is a probability density function, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{0}^{\infty} ce^{-2x}\,dx = 1$$

$$c\left(\frac{e^{-2x}}{-2}\right)_0^{\infty} = 1 \Rightarrow \frac{c}{2} = 1 \Rightarrow c = 2$$

$$P(X > 2) = \int_{2}^{\infty} f(x)\,dx = \int_{2}^{\infty} 2e^{-2x}\,dx = 2\left(\frac{e^{-2x}}{-2}\right)_2^{\infty} = e^{-4}$$

11.(b)(i) Describe the situations in which geometric distributions could be used. Obtain its moment generating function.

Solution:

Geometric distribution can be used, when the number of trials required to get first success. Let the random variable X denote the number of trials of a random experiment required to obtain the first success. Obviously X can assume the values $1, 2, 3, \ldots$

Now $X = r$iff the first $(r-1)$ trails result in failure and r^{th} trail results in success.

Then $P(X = r) = q^{r-1}p, r = 1,2, \ldots$

Then X is said to follow a geometric distribution.

Moment generating function is given by

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p = \frac{p}{q}\sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{q}\left((qe^t)^1 + (qe^t)^2 + (qe^t)^3 + \cdots\right)$$

$$= qe^t\frac{p}{q}\left(1 + qe^t + (qe^t)^2 + \cdots\right) = pe^t(1 - qe^t)^{-1}$$

$$M_X(t) = \frac{pe^t}{(1 - qe^t)}$$

11.(b)(ii) A coin having probability P of coming up heads is successively flipped until the r^{th} head appears. Argue that X, the number of flips required will be $n, n \geq r$ with probability

$$\boldsymbol{P[X = r] = \binom{n-1}{r-1} p^r q^{n-r}, \quad n \geq r}$$

Solution:

Suppose that n independent trail each of which result in a success with probability P. If X represents the number of success that occur in the n trails.

Then X is said to be a binomial random variable with parameter n and p is given by

$$P[X=r]=\binom{n}{r}p^r q^{n-r}, r=0,1,2,\dots,n$$

Let the first $(n-1)$ flips must have $(r-1)$ heads and then the n^{th} flip results head. These two events are independent.

$$P[X=r]=P[getting\ (r-1)\ heads\ in\ (n-1)\ trails]\times P[getting\ head\ in\ n^{th}\ trail]$$

$$P[X=r]=\binom{n-1}{r-1}p^{r-1}q^{n-r}p, \quad n\geq r$$

$$P[X=r]=\binom{n-1}{r-1}p^{r}q^{n-r}, \quad n\geq r$$

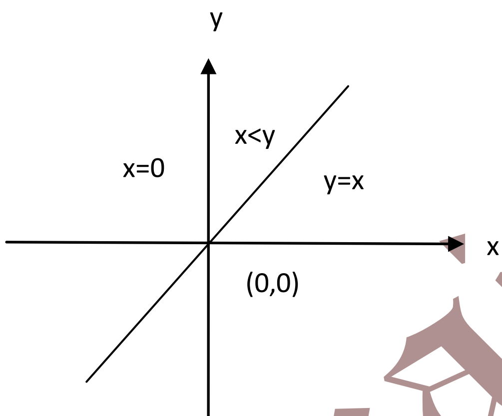
12.(a)(i) Suppose that X and Y are independent non-negative continuous random variable having densities $f_X(x)$ and $f_Y(y)$respectively. Compute $P[X<Y]$.

Solution:

Given X and Y are independent non-negative continuous random variable, then

$$f(x,y)=f_X(x)f_Y(y)\ldots(1)$$

$$P[X<Y]=\iint\limits_{X<Y} f(x,y)dxdy$$



The region is bounded by $x=0, y=x$ and $y=\infty$.

$$P[X<Y]=\iint f_X(x)f_Y(y)dxdy \quad from\ (1)$$

$$P[X<Y]=\int_0^{\infty}\int_0^{y} f_X(x)f_Y(y)dxdy$$

12.(a)(ii) The joint density of X and Y is given by

$$\boldsymbol{f(x,y)}=\begin{cases}\frac{1}{2}\boldsymbol{ye^{-xy}}, & \boldsymbol{0}<x<\infty, 0<y<2\\ \boldsymbol{0}, & \boldsymbol{otherwise}\end{cases}$$

Calculate the conditional density of X given $Y=1$.

Solution:

Conditional density of X given $Y=1$ is

$$f(X/Y=1)=\frac{f(x,y=1)}{f_Y(y)}$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx$$

$$f_Y(y)=\int_0^{\infty}\frac{1}{2}ye^{-xy}\,dx=\frac{1}{2}\left(\frac{ye^{-xy}}{-y}\right)_0^{\infty}=\frac{1}{2}$$

$$f(X/Y=1)=\frac{f(x,y=1)}{f_Y(y)}=\frac{\frac{1}{2}e^{-x}}{\frac{1}{2}}=e^{-x},0<x<\infty$$

12.(b)(i) If the correlation coefficient is 0, then we conclude that they are independent? Justify your answer, through an example. What about the converse?

Solution:

If the correlation coefficient is 0, then we cannot conclude that they are independent.

For example let X and Y be standard normal variate and $Y=X^2$.

i.e., $X\sim N(0,1)$ and $Y\sim N(0,1)$.

Then $E(X)=0,E(y)=0$

$$E(xy)=E(xx^2)=E(x^3)=0$$

$$[Since\ odd\ moments\ about\ the\ origin\ vanishes\ for\ N(0,1)]$$

$$r_{xy}=\frac{cov(x,y)}{\sigma_x\sigma_y}=\frac{E(xy)-E(x)E(y)}{\sigma_x\sigma_y}=0$$

But we know that $Y=X^2$, i.e., X and Y are dependent variables.

Hence if correlation coefficient is 0, we cannot conclude that they are independent.

But if the random variables are independent then the correlation coefficient is 0.

12.(b)(ii) Let X and Y be independent random variables both uniformly distributed on $(0,1)$. Calculate the probability density of $X+Y$.

Solution:

Let X and Y be independent random variables both uniformly distributed on (0,1).

$$f_X(x)=1\ ,0<x<1$$

$$f_Y(y)=1,0<y<1$$

Since X and Y are independent random variables

$$f(x,y)=f_X(x)f_Y(y)$$

$f(x,y)=1,0<x<1,0<y<1$

Let $U=X+Y\ and\ V=Y$

The joint pdf of (U,V) is given by

$$f(u,v)=|J|f(x,y)$$

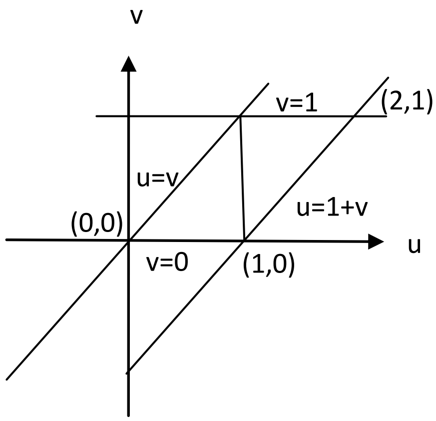
$$|J|=\begin{vmatrix}\frac{\partial X}{\partial U} & \frac{\partial Y}{\partial U}\\ \frac{\partial X}{\partial V} & \frac{\partial Y}{\partial V}\end{vmatrix}$$

$$U=X+Y\ and\ V=Y\Rightarrow\ X=U-V\ and\ Y=V$$

$$|J| = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f(u,v) = 1, 0 < v < 1, v < u < 1 + v$$

Pdf of U is given by



$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)dv = \begin{cases} \int_0^u dv, 0 < u < 1 \\ \int_{u-1}^{1} dv, 1 < u < 2 \end{cases}$$

$$f_U(u) = \begin{cases} u, 0 < u < 1 \\ 2 - u, 1 < u < 2 \end{cases}$$

13.(a)(i) Let the Markov Chain consisting of the states 0, 1, 2, 3 have the transition probability Matrix

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \frac{\mathbf{1}}{\mathbf{2}} & \frac{\mathbf{1}}{\mathbf{2}} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Determine which states are transient and which are recurrent by defining transient and recurrent states.

Solution:

Given

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

The state i is said to be transient if the return to state i is uncertain.

$$i.e., F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} < 1$$

The state i is said to be recurrent if the return to state i is certain.

$$i.e., F_{ii} = \sum_{n=1}^{\infty} f_{ii}^{(n)} = 1$$

$$P^2 = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$P^4 = P, P^5 = P^2, P^6 = P^3, \dots$$

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} = 0 + 0 + 1 = 1$$

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} + f_{11}^{(3)} = 0 + 0 + 1 = 1$$

$$F_{22} = \sum_{n=1}^{\infty} f_{22}^{(n)} = f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + f_{22}^{(4)} + f_{22}^{(5)} + f_{22}^{(6)} + \cdots = 0 + 0 + \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1$$

$F_{33} = \sum_{n=1}^{\infty} f_{33}^{(n)} = f_{33}^{(1)} + f_{33}^{(2)} + f_{33}^{(3)} + f_{33}^{(4)} + f_{33}^{(5)} + f_{33}^{(6)} + \cdots = 0 + 0 + \frac{1}{2} + 0 + 0 + \frac{1}{2} = 1$

which implies that the states $0, 1, 2, 3$ are recurrent.

Since it is a finite chain. All the states are recurrent.

13.(a)(ii) Suppose that whether or not it rains today depends on previous weather conditions through the last two days. Show how this system may be analyzed by using a Markov chain. How many states are needed?

Solution:

We can transform this model into a Markov chain by saying that the state at any time is determined by the weather conditions during both the day and the previous day.

In otherwords are can say that the process is in.

State 0 = if it rained both today and yesterday

State 1 = if it rained today but not yesterday

State 2 = if it rained yesterday but not today

State 3 = if it did not rain either yesterday or today

The preceding would then represent a four state Markov chain having transition probability matrix

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} P_1 & 0 & 1-P_1 & 0 \\ P_2 & 0 & 1-P_2 & 0 \\ 0 & P_3 & 0 & 1-P_3 \\ 0 & P_4 & 0 & 1-P_4 \end{bmatrix} \end{array}$$

Where P_1 is the probability that if it has rained for the past two days, then it will rain tomorrow.

P_2 is the probability that if it rained today but not yesterday.

P_3 is the probability that if it has rained yesterday but not today then it will rain tomorrow.

P_4 is the probability that if it has not rained in the past two days, then it will rain tomorrow.

Transforming the given process into Markov chain we observed that four states namely 0,1,2,3 are required for this problem.

13.(b)(i) Derive Chapman – Kolmogorov equations.

If P is the tpm of a homogeneous Markov chain then the n^{th} step tpm $P^{(n)} = P^n$

$$\left\{P_{ij}^{(n)}\right\} = \left\{P_{ij}\right\}^n$$

Proof:

Let P is a tpm of markov chain.

We know that,

$$P_{ij}(n-1,n) = P_{ij}(m-1,m)$$

Let $n = 1$

$$P_{ij}^{(1)} = P\{x_1 = j/x_0 = i\} = P_{ij} = P$$

Let $n = 2$

$$P_{ij}^{(2)} = P\{x_2 = j/x_0 = i\}$$

If in second step, the state j can be reached from state i through some intermediate state k.

$$P_{ij}^{(2)} = P\{x_2 = j/x_0 = i\}$$

$$= P\{x_2 = j, x_1 = k/x_0 = i\}$$

$$= P\{x_2 = j/x_1 = k, x_0 = i\}P\{x_1 = k/x_0 = i\}$$

$$= P\{x_2 = j/x_1 = k\}P\{x_1 = k/x_0 = i\}$$

$$= P_{kj}^{(1)}P_{ik}^{(1)}$$

$$P_{ij}^{(2)} = P_{ik}^{(1)}P_{kj}^{(1)}$$

The probability of j^{th} state can be reached from i^{th} state through the intermediate state $k = 1,2,3,\dots$

The k states are mutually exclusive

$$P_{ij}^{(2)} = \sum_k P_{ik}^{(1)}P_{kj}^{(1)}$$

$$P^{(2)} = P.P = P^2$$

i.e second step tpm = product of one step tpm

Consider 3rd step

$$P_{ij}^{(3)} = P\{x_3 = j/x_0 = i\}$$

$$= \sum_k P\{x_3 = j/x_2 = k\}P\{x_2 = k/x_0 = i\}$$

$$P_{ij}^{(3)} = \sum_k P_{kj}^{(1)}P_{ik}^{(2)} = \sum_k P_{ik}^{(2)}P_{kj}^{(1)} = P^2P = P^3$$

Similarly, $P^{(4)} = P^{(3)}P = P^3P = P^4$

In general, $P^{(n)} = P^n$

13.(b)(ii) Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

Solution:

The tpm of the above Markov chain, with state space=(Car, Truck) is

$$P = \begin{array}{c} \\ C \\ T \end{array}\begin{array}{c} \begin{array}{cc} C & T \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} \end{array}$$

$$\pi P = \pi \text{ and } \pi_1 + \pi_2 = 1 \ldots (1)$$

$$(\pi_1 \quad \pi_2)\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{1}{5} & \frac{4}{5} \end{bmatrix} = (\pi_1 \quad \pi_2)$$

$$\frac{1}{4}\pi_1 + \frac{1}{5}\pi_2 = \pi_1 \Rightarrow \frac{3}{4}\pi_1 = \frac{1}{5}\pi_2$$

$$\frac{3}{4}\pi_1 + \frac{4}{5}\pi_2 = \pi_2 \Rightarrow \frac{3}{4}\pi_1 = \frac{1}{5}\pi_2 \Rightarrow \pi_1 = \frac{4}{15}\pi_2$$

$$\pi_1 + \pi_2 = 1 \Rightarrow \frac{4}{15}\pi_2 + \pi_2 = 1 \Rightarrow \frac{19}{15}\pi_2 = 1$$

$$\pi_2 = \frac{15}{19}$$

$$from\ (1), we\ get\ \pi_1 = \frac{4}{19}$$

Which is the fraction of vehicles on the road are trucks.

14.(a) Define birth and death process. Obtain its steady state probabilities. How it could be used to
find the steady state solution for the M/M/1 model?

Solution:

If $X(t)$ represents the number of individuals present at time t in a population in which two types of events occur – one representing birth which contributes to its increase and the other representing death which contributes to its decrease, then the discrete random process $\{X(t)\}$

is called the birth and death process, provided the two events, birth and death are governed by the following postulates:

If $X(t) = n \qquad (n > 0)$

(i) P[1 birth in $(t, t + \Delta t)] = \lambda_n \Delta t + O(\Delta t)$

(ii) P[0 birth in $(t, t + \Delta t)] = 1 - \lambda_n \Delta t + O(\Delta t)$

(iii) P[2 or more births in $(t, t + \Delta t)] = O(\Delta t)$

(iv) Births occurring in $(t, t + \Delta t)$ are independent of time since last birth.

(v) P[1 death in $(t, t + \Delta t)] = \mu_n \Delta t + O(\Delta t)$

(vi) P[0 death in $(t, t + \Delta t)] = 1 - \mu_n \Delta t + O(\Delta t)$

(vii) P[2 or more deaths in $(t, t + \Delta t)] = O(\Delta t)$

(viii) Death occurring in $(t, t + \Delta t)$ are independent of time since last death.

(ix) The birth and death occur independently of each other at any time.

Let n be the size of the population at time t and $P_n(t) = P\{X(t) = n\}$ be the corresponding probability. Then we have $P_n(t + \Delta t) = P\{X(t + \Delta t) = n\}$ is the probability that the size is n at time $t + \Delta t$. Now the event $X(t + \Delta t) = n$ can occur in any one of the given four mutually exclusive ways.

(a) $X(t) = n$ and no birth or death in $(t, t + \Delta t)$

(b) $X(t) = n + 1$ and no birth and one death in $(t, t + \Delta t)$

(c) $X(t) = n - 1$ and one birth and no death in $(t, t + \Delta t)$

(d) $X(t) = n$ and one birth and one death in $(t, t + \Delta t)$

Hence we have

$P_n(t + \Delta t) =$ sum of probabilities of the above four ways

$$P_n(t + \Delta t) = P(a) + P(b) + P(c) + P(d)$$

$$\begin{aligned} P_n(t + \Delta t) = {} & P_n(t)(1 - \lambda_n \Delta t)(1 - \mu_n \Delta t) + P_{n+1}(t)(1 - \lambda_{n+1} \Delta t)\mu_{n+1} \Delta t \\ & + P_{n-1}(t)(\lambda_{n-1} \Delta t)(1 - \mu_{n-1} \Delta t) + P_n(t)(\lambda_n \Delta t)(\mu_n \Delta t) \end{aligned}$$

$$
\begin{aligned}
P_n(t+\Delta t) = {} & P_n(t) - P_n(t)(\lambda_n \Delta t) - P_n(t)(\mu_n \Delta t) + P_n(t)(\lambda_n \Delta t)(\mu_n \Delta t) + P_{n+1}(t)\mu_{n+1}\Delta t \\
& - P_{n+1}(t)\mu_{n+1}\Delta t \lambda_{n+1}\Delta t + P_{n-1}(t)(\lambda_{n-1}\Delta t) - P_{n-1}(t)(\lambda_{n-1}\Delta t)(\mu_{n-1}\Delta t) \\
& + P_n(t)(\lambda_n \Delta t)(\mu_n \Delta t)
\end{aligned}
$$

$$
\begin{aligned}
P_n(t+\Delta t) = {} & P_n(t) - P_n(t)(\lambda_n \Delta t) - P_n(t)(\mu_n \Delta t) + P_n(t)\mu_n \lambda_n (\Delta t)^2 + P_{n+1}(t)\mu_{n+1}\Delta t \\
& - P_{n+1}(t)\mu_{n+1}\lambda_{n+1}(\Delta t)^2 + P_{n-1}(t)(\lambda_{n-1}\Delta t) - P_{n-1}(t)(\lambda_{n-1})(\mu_{n-1})(\Delta t)^2 \\
& + P_n(t)(\lambda_n)(\mu_n)(\Delta t)^2 \ldots (1)
\end{aligned}
$$

Omitting terms containing $(\Delta t)^2$ in (1), we get

$$P_n(t+\Delta t) = P_n(t) - P_n(t)(\lambda_n \Delta t) - P_n(t)(\mu_n \Delta t) + P_{n+1}(t)\mu_{n+1}\Delta t + P_{n-1}(t)(\lambda_{n-1}\Delta t)$$

$$P_n(t+\Delta t) - P_n(t) = -P_n(t)(\lambda_n \Delta t) - P_n(t)(\mu_n \Delta t) + P_{n+1}(t)\mu_{n+1}\Delta t + P_{n-1}(t)(\lambda_{n-1}\Delta t)$$

$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -P_n(t)\lambda_n - P_n(t)\mu_n + P_{n+1}(t)\mu_{n+1} + P_{n-1}(t)\lambda_{n-1}$$

Taking limit on both sides of (1) as $\Delta t \to 0$, we get

$$P_n'(t) = \lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)\, P_n(t) + \mu_{n+1}P_{n+1}(t) \ldots (2)\ for\ n \neq 0$$

For $n = 0$ no death is possible in the interval $(t, t+\Delta t)$ and $X(t) = n - 1 = -1$ is meaningless.

$$\therefore\ P_0'(t) = -\lambda_0\, P_0(t) + \mu_1 P_1(t) \ldots (3)$$

In the steady state, $P_n(t)$ and $P_0(t)$ are independent of time and hence $P_n'(t) = 0$ and $P_0'(t) = 0$

Therefore (2) and (3) becomes

$$\lambda_{n-1}P_{n-1}(t) - (\lambda_n + \mu_n)\, P_n(t) + \mu_{n+1}P_{n+1}(t) = 0 \ldots (4)$$

$$-\lambda_0\, P_0(t) + \mu_1 P_1(t) = 0 \ldots (5)$$

Values of P_0 and P_n for Poisson Queue systems:

From (5), we get

$$P_1 = \frac{\lambda_0}{\mu_1} P_0 \ldots (6)$$

Put $n = 1$ in (4) , we get

$$\lambda_0 P_0 - (\lambda_1 + \mu_1)\, P_1 + \mu_2 P_2 = 0$$

$$\mu_2 P_2 = -\lambda_0 P_0 + (\lambda_1 + \mu_1)\, P_1$$

$$\mu_2 P_2 = -\lambda_0 P_0 + (\lambda_1 + \mu_1)\frac{\lambda_0}{\mu_1} P_0$$

$$\mu_2 P_2 = -\lambda_0 P_0 + \frac{\lambda_0 \lambda_1}{\mu_1} P_0 + -\lambda_0 P_0$$

$$\mu_2 P_2 = \frac{\lambda_0 \lambda_1}{\mu_1} P_0 \Rightarrow P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

In general,

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0, for\ n = 1,2, \ldots \qquad \ldots (7)$$

$$We\ know\ that\ \sum_{n=0}^{\infty} P_n = 1$$

$$P_0 + \sum_{n=1}^{\infty} P_n = 1$$

$$P_0 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0 = 1$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}} \ldots (8)$$

Steady state solution for the M/M/1 model:

Put $\lambda_n = \lambda$, $\mu_n = \mu$ in (7) and (8), we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots}$$

$$P_0 = \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{-1}} = 1 - \frac{\lambda}{\mu}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

14.(b) Calculate any four measures of effectiveness of M/M/1 queueing model.

Solution:

The Values of P_0 and P_n for Poisson Queue systems:

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n}}$$

$$P_n = \frac{\lambda_0 \lambda_1 \ldots \lambda_{n-1}}{\mu_1 \mu_2 \ldots \mu_n} P_0, n \geq 1$$

Steady state solution for the M/M/1 model:

Put $\lambda_n = \lambda,\ \mu_n = \mu$ in (7) and (8), we get

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots}$$

$$P_0 = \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{-1}} = 1 - \frac{\lambda}{\mu}$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

Average number of customers in the system:

$$L_s = E(n) = \sum_{n=0}^{\infty} nP_n$$

$$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$= \left(1 - \frac{\lambda}{\mu}\right)\left(0 + 1\left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \cdots\right)$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left(1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\cdots\right)$$

$$= \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2} = \left(\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-1}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\left(\frac{\lambda}{\mu}\right)}{\left(\frac{\mu-\lambda}{\mu}\right)} = \frac{\lambda}{\mu-\lambda}$$

$$L_s = \frac{\lambda}{\mu-\lambda}$$

Average number of customers in the queue:

$$L_q = E(n-1) = \sum_{n=2}^{\infty}(n-1)P_n$$

$$= \sum_{n=2}^{\infty}(n-1)\left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right)^2+2\left(\frac{\lambda}{\mu}\right)^3+3\left(\frac{\lambda}{\mu}\right)^4+\cdots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\left[1+2\left(\frac{\lambda}{\mu}\right)+3\left(\frac{\lambda}{\mu}\right)^2+\cdots\right]$$

$$= \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)\left(1-\frac{\lambda}{\mu}\right)^{-2} = \left(\frac{\lambda}{\mu}\right)^2\left(1-\frac{\lambda}{\mu}\right)^{-1}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^2}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{\left(\frac{\mu-\lambda}{\mu}\right)} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

Average waiting time in the system:

$$W_s = \frac{L_s}{\lambda} = \frac{\lambda}{\lambda(\mu-\lambda)} = \frac{1}{(\mu-\lambda)}$$

Average waiting time in the queue:

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda^2}{\lambda\mu(\mu-\lambda)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

15. (a) Derive the expected steady state system size for the single server queues with Poisson input and General service or Derive Pollaczek-Khintchine formula.

Solution:

Let n and n_1 be the number of customer in the system at times t and $t+T$, when two consecutive customers have just left the system after getting service.

Let $f(t), E(T)$ and $var(T)$ be the probability density function, mean and variance of T. Let k be the number of customers arriving in the system during the service time T.

Hence

$$n_1 = \begin{cases} k, & if\ n = 0 \\ n-1+k, & if\ n > 0 \end{cases}$$

Where $k = 0,1,2,3, \ldots$, is the number of arrivals during the service time. If

$$\delta = \begin{cases} 1 & if\ n = 0 \\ 0 & if\ n > 1 \end{cases}$$

Then $n_1 = n - 1 + \delta + k$ … (1)

$$E(n_1) = E(n-1+\delta+k) \Rightarrow E(n_1) = E(n) - 1 + E(\delta) + E(k) \quad \ldots (2)$$

When the system has reached the steady state, the probability of the number of customers in the system will be constant. Hence

$$E(n_1) = E(n)\ and\ E(n_1^2) = E(n^2) \quad \ldots (3)$$

Substituting (3) in (2), we get

$$-1 + E(\delta) + E(k) = 0 \Rightarrow E(\delta) = 1 - E(k) \quad \ldots (4)$$

$$n_1^2 = (n+k-1+\delta)^2$$

$$n_1^2 = n^2 + (k-1)^2 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta(k-1)$$

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta k - 2\delta \ldots (5)$$

Since $\delta = \delta^2$ and n $\delta = 0$,we get

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta + 2n(k-1) + 2\delta k - 2\delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + 2\delta k - \delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)$$

$$E\big(2n(1-k)\big) = E\big(n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)\big)$$

$$2E(n)E(1-k) = E(n^2) - E(n_1^2) + E(k^2) - 2E(k) + 1 + E(\delta)E(2k-1)$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)E(2k-1) \quad \text{(Using (3)\& (4))}$$

$$2E(n)\big(1-E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)(2E(k)-1)$$

$$E(n) = \frac{\big(E(k^2) - 2E(k) + 1 + \big(1-E(k)\big)(2E(k)-1)\big)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-1 + 1 - E(k)\big)(2E(k)-1)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-E(k)\big)(2E(k)-1)}{2\big(1-E(k)\big)} = \frac{E(k^2) + E(k) - 2E^2(k)}{2\big(1-E(k)\big)}$$

$$E(n) = \frac{E(k^2) - E^2(k) + E(k) - E^2(k)}{2\big(1-E(k)\big)} = \frac{Var\,(k) + E(k) - E^2(k)}{2\big(1-E(k)\big)} \ldots (6)$$

Since the number k of arrivals follows Poisson process with parameter λ.

$$E(k/T) = \lambda T$$

$$E(k^2/T) = \lambda^2 T^2 + \lambda T$$

$$E(k) = \int_0^\infty E(k/T)\, f(t)dt = \lambda \int_0^\infty T\, f(t)dt = \lambda E(T) \ldots (7)$$

$$E(k^2) = \int_0^\infty E(k^2/T)\, f(t)dt = \int_0^\infty (\lambda^2 T^2 + \lambda T)\, f(t)dt = \lambda^2 \int_0^\infty T^2\, f(t)dt + \lambda \int_0^\infty T\, f(t)dt$$

$$E(k^2) = \lambda^2 E(T^2) + \lambda E(T) \ldots (8)$$

$$Var(k) = E(k^2) - E^2(k) = \lambda^2 E(T^2) + \lambda E(T) - \lambda^2 E^2(T) = \lambda^2\big(E(T^2) - E^2(T)\big) + \lambda E(T)$$

$$Var(k) = \lambda^2 Var(T) + \lambda E(T) \ldots (9)$$

Substituting (7),(8) and (9) in (6), we get

$$E(n) = \frac{\lambda^2 Var(T) + \lambda E(T) + \lambda E(T) - \lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T) - 2\lambda^2 E^2(T)}{2\big(1 - \lambda E(T)\big)}$$

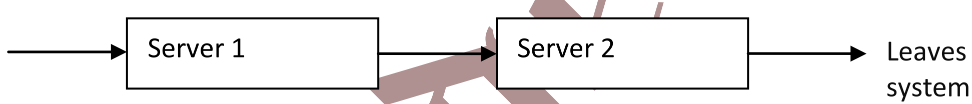
$$E(n) = \frac{\lambda^2\big(Var(T) + E^2(T)\big) + 2\lambda E(T)\big(1 - \lambda E(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

$$E(n) = \lambda E(T) + \frac{\lambda^2\big(Var(T) + E^2(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

15. (b) Explain how queueing theory could be used to study computer networks.

Solution:

Consider a two server system in which customers arrive at a Poisson rate λ at server 1. After being served by server 1, they join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server serves only one customer at a time with server I taking exponential time with rate μ_i such a system is called a sequential system.



Consider a system of k servers customers arrive from outside the system to server i. $i = 1,2,\dots,k$ in accordance with independent Poisson process with rate r_i, they join the queue at i until their turn comes one a customer is served by server i, he then joins the queue in front of the server j, with probability P_{ij} or leaves the system with probability P_{i0}. Hence if we let λ_j be the total arrival rate of customers to server j, then

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$$

which is called flow balance equation.

A queueing network of k nodes is called a closed Jackson network if new customers never enter into and the existing customer never depart from the system i.e $r_i = 0$ and $P_{i0} = 0$ for all i.

The flow balance equation of this model becomes

$$\lambda_j = \sum_{i=1}^{k} \lambda_i P_{ij}$$

Jackson’s open network concept can be extended when the nodes are multi server nodes.

In this case the network behaves as if each node is an independent $M/M/S$ model.