November/ December 2011

Part-A

1. A continuous random variable X that can assume any value between $x = 2$ and $x = 5$ has a density function given by $f(x) = k(1 + x)$. Find $(X < 4)$.

Solution:

$$We\ know\ that \int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_{2}^{5} k(1 + x)dx = 1$$

$$k\left[x + \frac{x^2}{2}\right]_2^5 = 1 \Rightarrow k\left[5 + \frac{25}{2} - 2 - 2\right] = 1 \Rightarrow \frac{27}{2}k = 1 \Rightarrow k = \frac{2}{27}$$

$$p(X < 4) = \int_{2}^{4} \frac{2}{27}(1 + x)dx = \frac{2}{27}\left(x + \frac{x^2}{2}\right)_2^4 = \frac{2}{27}(4 + 8 - 2 - 2) = \frac{16}{27}$$

2. Give the probability law of Poisson distribution and also its mean and variance.

Solution:

The probability law of Poisson distribution is given by

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \lambda > 0, x = 1,2,3, \ldots$$

$$Mean = \lambda, Variance = \lambda$$

3. *The joint pdf of the random variable* (X, Y) *is given by* $f(x, y) = kxye^{-(x^2+y^2)}, x > 0,$ $y > 0$. *Find the value of* k.

Solution:

$$We\ know\ that \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} kxye^{-(x^2+y^2)}dxdy = 1 \Rightarrow k\int_{0}^{\infty}\int_{0}^{\infty} xye^{-x^2}e^{-y^2}dxdy = 1 \Rightarrow k\int_{0}^{\infty}\left[\frac{1}{2}\right]ye^{-y^2}dy = 1$$

$$\Rightarrow \frac{k}{4} = 1 \Rightarrow k = 4$$

4. *Given the random variable* X *with density function* $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & elsewhere \end{cases}$.

Find the pdf of $Y = 8X^3$.

Solution:

Let $f(y)$ be the probability density function of the random variable Y.

$$y = 8x^3 \Rightarrow x^3 = \frac{y}{8} \Rightarrow x = \frac{1}{2}y^{\frac{1}{3}}$$

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{3}\right)y^{\frac{1}{3}-1} = \frac{1}{6}y^{-\left(\frac{2}{3}\right)}$$

$$f(x) = 2x = y^{\frac{1}{3}}$$

$$x > 0 \Rightarrow \frac{1}{2}y^{\frac{1}{3}} > 0 \Rightarrow y > 0$$

$$x < 1 \Rightarrow \frac{1}{2}y^{\frac{1}{3}} < 1 \Rightarrow y^{\frac{1}{3}} < 2 \Rightarrow y < 2^3 \Rightarrow y < 8$$

$$f(y) = \left|\frac{dx}{dy}\right| f(x) = \frac{1}{6}y^{-\left(\frac{2}{3}\right)}\left(y^{\frac{1}{3}}\right) = \frac{1}{6}y^{-\frac{1}{3}} , \qquad 0 < y < 8$$

$$f(y) = \frac{1}{6}y^{-\frac{1}{3}}, \qquad 0 < y < 8$$

5. Define transition probability matrix.

Solution:

Let P_{ij} be the transition probabilities of the markov chain. The elements are written in the matrix form as

$$P = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{pmatrix}$$

The above matrix P is called TPM of the markov chain as P is stochastic matrix.

6. Define markov process.

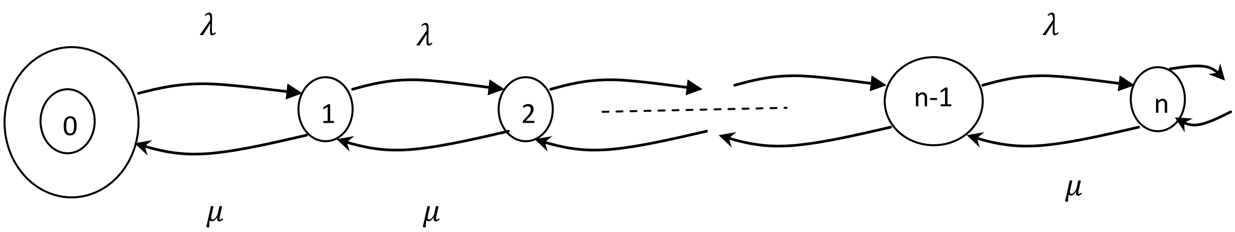
Solution:

A random process $\{X(t)\}$ is called a markov process if, for $(t_1 < t_2 < \cdots < t_n)$ we have

$$P\{X(t_n) = a_n / X(t_{n-1}) = a_{n-1}, X(t_{n-2}) = a_{n-2}, \dots, X(t_1) = a_1\}$$
$$= P\{X(t_n) = a_n / X(t_{n-1}) = a_{n-1}\}.$$

7. Draw the state transition diagram for $M/M/1$ queueing model.

Solution:



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8. What do the letters in the symbolic representation $(a/b/c):(d/e)$ of a queueing model represent?

Solution:

a — Probability distribution of number of arrivals

b — Probability distribution of service time

c — Number of services

d —Capacity of the system

e — Service discipline

9. Write the Pollaczek-Khintchine formula.

Solution:

Pollaczek-Khintchine formula for $M/G/1$ is

$$L_s = \lambda E(T) + \frac{\lambda^2\big(V(T) + E^2(T)\big)}{2\big(1 - \lambda E(T)\big)}$$

10. Define Series Queues.

A special type of open queueing network called series queue.

In open network there are a series of service facilities which each customer should visit in the given order before leaving the system. The nodes form a series system flow always in a single direction from node to node. Customers enter from outside only at node 1 and depart only from node k.

Example: Registration process in university, in clinic physical examination procedure.

Part-B

$\boldsymbol{11. (a)\ (i)\ The\ distribution\ function\ of\ a\ continuous\ random\ variable\ X\ is\ given\ by}$

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le \frac{1}{2} \\ 1 - \frac{3}{25}(3 - x)^2, & \frac{1}{2} \le x \le 3 \\ 1, & x > 3 \end{cases}$$

$\boldsymbol{Find\ the\ pdf\ of\ X\ and\ evaluate\ P(|X| \le 1)\ and\ P\left(\frac{1}{3} < X < 4\right) using\ both\ cdf\ and\ pdf.}$

Solution:

$$f(x) = F'(x)$$

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$$f(x) = \begin{cases} 0, & x < 0 \\ 2x\,, & 0 \le x \le \frac{1}{2} \\ \frac{6}{25}(3-x), & \frac{1}{2} \le x \le 3 \\ 0, & x > 3 \end{cases}$$

Using pdf

$$P(|X| \le 1) = P(-1 \le X \le 1) = \int_{-1}^{1} f(x)dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{\frac{1}{2}} 2x\,dx + \int_{\frac{1}{2}}^{1} \frac{6}{25}(3-x)\,dx = 2\left(\frac{x^2}{2}\right)_0^{\frac{1}{2}} + \frac{6}{25}\left(3x - \frac{x^2}{2}\right)_{\frac{1}{2}}^{1}$$

$$= \frac{1}{4} + \frac{6}{25}\left(3 - \frac{1}{2} - \frac{3}{2} + \frac{1}{8}\right) = \frac{1}{4} + \frac{6}{25}\left(\frac{9}{8}\right) = 0.25 + 0.27 = 0.52$$

$$P(|X| \le 1) = 0.52$$

$$P\left(\frac{1}{3} < X < 4\right) = \int_{\frac{1}{3}}^{4} f(x)dx = \int_{\frac{1}{3}}^{\frac{1}{2}} 2xdx + \int_{\frac{1}{2}}^{3} \frac{6}{25}(3-x)dx + \int_{3}^{4} 0dx$$

$$P\left(\frac{1}{3} < X < 4\right) = 2\left(\frac{x^2}{2}\right)_{\frac{1}{3}}^{\frac{1}{2}} + \frac{6}{25}\left(3x - \frac{x^2}{2}\right)_{\frac{1}{2}}^{3} = \frac{1}{4} - \frac{1}{9} + \frac{6}{25}\left(9 - \frac{9}{2} - \frac{3}{2} + \frac{1}{8}\right) = \frac{8}{9}$$

Using cdf

$$P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) = 1 - \frac{3}{25}(3-1)^2 - 0 = 1 - \frac{12}{25} = 0.52$$

$$P\left(\frac{1}{3} < X < 4\right) = F(4) - F\left(\frac{1}{3}\right) = 1 - \left(\frac{1}{3}\right)^2 = \frac{8}{9}$$

11. (a) (ii) A random variable X has the following probability distribution

x	-2	-1	0	1	2	3
$P(x)$	0.1	k	0.2	$2k$	0.3	$3k$

(1) Find k, (2) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$, (3) find the cumulative distribution function of X and (4) evaluate the mean of X.

Solution:

$(1)\ We\ know\ that\ \sum P(x) = 1$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1 \Rightarrow 6k + 0.6 = 1 \Rightarrow 6k = 0.4 \Rightarrow k = \frac{0.4}{6} = 0.067$$

(2) $P(X < 2) = P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$

$P(X < 2) = 0.1 + 0.067 + 0.2 + 0.134 = 0.501$

$P(-2 < X < 2) = P(X = -1) + P(X = 0) + P(X = 1) = 0.067 + 0.2 + 0.134 = 0.401$

(3) The cumulative distribution function of X

x	-2	-1	0	1	2	3
$P(x)$	0.1	0.167	0.367	0.501	0.801	1

(4) $E[x] = \sum xP(x) = -2(0.1) - 1(0.067) + 0(0.2) + 1(0.134) + 2(0.3) + 3(0.201)$

$E[x] = -0.2 - 0.067 + 0 + 0.134 + 0.6 + 0.603 = 1.07$

$\boldsymbol{(b)(i) The\ probability\ function\ of\ an\ infinite\ discrete\ distribution\ is\ given\ by}$

$\boldsymbol{P(X = j) = \frac{1}{2^j}, j = 1, 2, \ldots, \infty. Verify\ that\ the\ total\ probability\ is\ 1\ and\ find\ the}$

$\boldsymbol{mean\ and\ variance\ of\ the\ distribution. Find\ also\ P(X\ is\ even), P(X \geq 5)\ and}$

$\boldsymbol{P(X\ is\ divisible\ by\ 3).}$

Solution:

$$\sum_{j=1}^{\infty} P(X = j) = \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-1}$$

$$\sum_{j=1}^{\infty} P(X = j) = 1$$

$$E[X] = \sum_{j=1}^{\infty} jP(X = j) = \sum_{j=1}^{\infty} j\frac{1}{2^j} = \frac{1}{2} + 2\frac{1}{2^2} + 3\frac{1}{2^3} + \cdots = \frac{1}{2}\left(1 + 2\ \frac{1}{2} + 3\ \frac{1}{2^2} + 4\ \frac{1}{2^3} + \cdots\right)$$

$$E[X] = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-2} = \frac{1}{2}\left(\frac{1}{2}\right)^{-2} = 2$$

$$E[X^2] = \sum_{j=1}^{\infty} j^2P(X = j) = \sum_{j=1}^{\infty} j^2\frac{1}{2^j} = \sum_{j=1}^{\infty} (j(j-1) + j)\frac{1}{2^j} = \sum_{j=1}^{\infty} j(j-1)\frac{1}{2^j} + \sum_{j=1}^{\infty} jP(X = j)$$

$$= \left(1.2\ \frac{1}{2^2} + 2.3\ \frac{1}{2^3} + 3.4\frac{1}{2^4} + \cdots\right) + 2 = \left(\frac{1}{2} + 1.3\frac{1}{2^2} + 2.3\frac{1}{2^3} + \cdots\right) + 2$$

$$E[X^2] = \frac{1}{2}\left(1 + 3\frac{1}{2} + 2.3\frac{1}{2^2} + \cdots\right) + 2 = \frac{1}{2}\left(1 - \frac{1}{2}\right)^{-3} + 2 = \frac{1}{2}\left(\frac{1}{2}\right)^{-3} + 2 = 6$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = 6 - 2^2 = 2$$

$$P(X\ is\ even) = P(X = 2) + P(X = 4) + P(X = 6) + \cdots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots = \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \cdots = \frac{1}{2^2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \cdots\right]$$

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$$= \frac{1}{2^2}\left(1 - \frac{1}{2^2}\right)^{-1} = \frac{1}{3}$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) + P(X = 7) + \cdots = \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \cdots$$

$$= \frac{1}{2^5}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = \frac{1}{2^5}\left(1 - \frac{1}{2}\right)^{-1} = \frac{1}{16}$$

$$P(X \text{ is divisible by } 3) = P(X = 3) + P(X = 6) + P(X = 9) + \cdots = \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots$$

$$= \frac{1}{2^3}\left(1 + \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots\right) = \frac{1}{2^3}\left[1 + \frac{1}{2^3} + \left(\frac{1}{2^3}\right)^2 + \left(\frac{1}{2^3}\right)^3 + \cdots\right] = \frac{1}{2^3}\left(1 - \frac{1}{2^3}\right)^{-1} = \frac{1}{7}$$

(ii) Define Gamma distribution and find its mean and variance.

Solution:

Let X be a continuous random variable with pdf

$$f(x) = \frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}, 0 \leq x \leq \infty, \lambda > 0$$

then, X is said to be gamma distribution with parameter λ.

$$E[x] = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{\infty} x\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}dx = \int_0^{\infty}\frac{e^{-x}x^{\lambda}}{\Gamma\lambda}dx = \int_0^{\infty}\frac{e^{-x}x^{(\lambda+1)-1}}{\Gamma\lambda}dx = \frac{\Gamma(\lambda+1)}{\Gamma\lambda} = \frac{\lambda\Gamma\lambda}{\Gamma\lambda}$$

$$= \lambda$$

$Mean = \lambda$ $\qquad\left[\because \int_0^{\infty} e^{-x}x^{n-1}dx = \Gamma n\right]$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2\frac{e^{-x}x^{\lambda-1}}{\Gamma\lambda}dx = \int_0^{\infty}\frac{e^{-x}x^{\lambda+1}}{\Gamma\lambda}dx = \int_0^{\infty}\frac{e^{-x}x^{(\lambda+2)-1}}{\Gamma\lambda}dx = \frac{\Gamma(\lambda+2)}{\Gamma\lambda}$$

$$\frac{\Gamma((\lambda+1)+1)}{\Gamma\lambda} = \frac{(\lambda+1)\Gamma(\lambda+1)}{\Gamma\lambda} = \frac{(\lambda+1)\lambda\Gamma\lambda}{\Gamma\lambda} = \lambda^2 + \lambda$$

$$Var(x) = E[x^2] - (E[x])^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

12. (a)(i) *The joint probability mass function of* (X, Y) *is given by* $P(x,y) = k(2x + 3y), x = 0,1,2; y = 1,2,3$. *Find all the marginal and conditional probability distributions.*

Solution:

$X\backslash Y$	1	2	3	$P_{i.}$
0	$3k$	$6k$	$9k$	$18k$
1	$5k$	$8k$	$11k$	$24k$
2	$7k$	$10k$	$13k$	$30k$
$P_{.j}$	$15k$	$24k$	$33k$	$72k$

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$$\sum P_{ij} = 1 \Rightarrow 72k = 1 \Rightarrow k = \frac{1}{72}$$

Marginal density function of X is given by

X	0	1	2
$P_{i.}$	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$

Marginal density function of Y is given by

Y	1	2	3
$P_{.j}$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$

Conditional probability distribution of X/Y is given by

$$P(X/Y) = \frac{P(x,y)}{P_{.j}}$$

$X\backslash Y$	1	2	3
0	$\frac{3}{15}$	$\frac{6}{24}$	$\frac{9}{33}$
1	$\frac{5}{15}$	$\frac{8}{24}$	$\frac{11}{33}$
2	$\frac{7}{15}$	$\frac{10}{24}$	$\frac{13}{33}$

Conditional probability distribution of Y/X is given by

$$P(Y/X) = \frac{P(x,y)}{P_{i.}}$$

$X\backslash Y$	1	2	3
0	$\frac{3}{18}$	$\frac{6}{18}$	$\frac{9}{18}$
1	$\frac{5}{24}$	$\frac{8}{24}$	$\frac{11}{24}$
2	$\frac{7}{30}$	$\frac{10}{30}$	$\frac{13}{30}$

(ii) State and prove central limit theorem.

Statement:

If $X_1, X_2, X_3, \dots, X_n, \dots$ be a sequence of independent identically distributed random variables with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma^2, i = 1,2,3, \dots, n, \dots$ and $S_n = X_1 + X_2 + X_3 + \cdots + X_n$, then under

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certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as $n \to \infty$.

$i.e., S_n \sim N(n\mu, n\sigma^2)$ as $n \to \infty$.

Proof:

Let $\{X_i\}, i = 1,2, \ldots$ be a sequence of independent binomially distributed random variable with parameter n and p, such that

$E[X_i] = \mu_i$ and $Var(X_i) = \sigma^2$

Let $S_n = X_1 + X_2 + X_3 + \cdots + X_n$ be a random variable.

The MGF of S_n is

$$M_{S_n}(t) = M_{X_1+X_2+X_3+\cdots+X_n}(t)$$

$$= M_{X_1}(t).M_{X_2}(t).M_{X_3}(t)\ldots M_{X_n}(t) = \left[M_{X_i}(t)\right]^n = (q+pe^t)^n$$

$$Let\ z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\ then\ the\ MGF\ of\ z\ is$$

$$M_z(t) = M_{\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)}(t) = M_{\left(\frac{\sqrt{n}\bar{X}-\sqrt{n}\mu}{\sigma}\right)}(t) = E\left[e^{\left(\frac{\sqrt{n}\bar{X}-\sqrt{n}\mu}{\sigma}\right)t}\right]$$

$$= E\left[e^{\left(\frac{\sqrt{n}\bar{X}}{\sigma}\right)t}e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}\right] = e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}E\left[e^{\left(\frac{\sqrt{n}\bar{X}}{\sigma}\right)t}\right]$$

$$= e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}M_{\bar{X}}\left(\frac{t\sqrt{n}}{\sigma}\right)$$

$$= e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}M_{\left(\frac{X_1+X_2+X_3+\cdots+X_n}{n}\right)}\left(\frac{t\sqrt{n}}{\sigma}\right)$$

$$= e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}M_{X_1+X_2+X_3+\cdots+X_n}\left(\frac{t\sqrt{n}}{\sigma n}\right) = e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}M_{X_1+X_2+X_3+\cdots+X_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$M_z(t) = e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

Taking log on both sides

$$\log M_z(t) = \log\left[e^{\left(\frac{-\sqrt{n}\mu}{\sigma}\right)t}\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[M_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[E\left(e^{\frac{tx}{\sigma\sqrt{n}}}\right)\right]$$

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$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[E\left(1 + \frac{\left(\frac{tx}{\sigma\sqrt{n}}\right)}{1!} + \frac{\left(\frac{tx}{\sigma\sqrt{n}}\right)^2}{2!} + \cdots\right)\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[E(1) + E\left(\frac{\left(\frac{tx}{\sigma\sqrt{n}}\right)}{1!}\right) + E\left(\frac{\left(\frac{tx}{\sigma\sqrt{n}}\right)^2}{2!}\right)\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[1 + \frac{t}{\sigma\sqrt{n}}E(x) + \frac{\left(\frac{t}{\sigma\sqrt{n}}\right)^2}{2!}E(x^2) + \cdots\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\log\left[1 + \frac{t}{\sigma\sqrt{n}}\mu_1' + \frac{t^2}{2\sigma^2 n}\mu_2' + \cdots\right]$$

$$= \frac{-\sqrt{n}\mu t}{\sigma} + n\left[\left(\frac{t}{\sigma\sqrt{n}}\mu_1' + \frac{t^2}{2\sigma^2 n}\mu_2' + \cdots\right) - \frac{1}{2}\left(\frac{t}{\sigma\sqrt{n}}\mu_1' + \frac{t^2}{2\sigma^2 n}\mu_2' + \cdots\right)^2 + \cdots\right]$$

$$\log M_z(t) = \frac{t^2}{2} + o\left(\frac{1}{\sqrt{n}}\right)$$

As $n \to \infty$, we have

$$\log M_z(t) \to \frac{t^2}{2}$$

$$\therefore M_z(t) \to e^{\frac{t^2}{2}}$$

$e^{\frac{t^2}{2}}$ *is the MGF of a standard normal variate.*

12. (b)(i) *If X and Y are independent random variables with pdf's* $\boldsymbol{e^{-x}, x \geq 0}$, ***and*** $\boldsymbol{e^{-y}, y \geq 0}$, ***respectively, find the density functions of*** $\boldsymbol{U = \frac{X}{X+Y}}$ ***and*** $\boldsymbol{V = X + Y}$. ***Are U and V independent?***

Solution:

Given $f_X(x) = e^{-x}, x \geq 0, f_Y(y) = e^{-y}, y \geq 0$

$$f(x,y) = f_X(x)f_Y(y) = e^{-(x+y)}, x \geq 0, y \geq 0 \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$u = \frac{x}{x+y}, v = x + y \Rightarrow u = \frac{x}{v} \Rightarrow x = uv, y = v - x = v - uv$$

$$x \geq 0 \Rightarrow uv \geq 0 \Rightarrow u \geq 0 \text{ or } v \geq 0$$

$$y \geq 0 \Rightarrow v - uv \geq 0 \Rightarrow v \geq uv \Rightarrow u \leq 1$$

$$x = uv, y = v - uv$$

$$\frac{\partial x}{\partial u} = v, \frac{\partial y}{\partial u} = -v, \frac{\partial x}{\partial v} = u, \frac{\partial y}{\partial v} = 1 - u$$

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$$|J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & -v \\ u & 1-u \end{vmatrix} = v - uv + uv = v$$

$$g(u,v) = |J| f(x,y) = ve^{-v}, 0 \le u \le 1, v \ge 0$$

$$g_U(u) = \int_{-\infty}^{\infty} g(u,v)dv = \int_0^{\infty} ve^{-v}dv = \left[v\left(\frac{e^{-v}}{-1}\right) - 1(e^{-v}) \right]_0^{\infty} = 1, 0 \le u \le 1$$

$$g_V(v) = \int_{-\infty}^{\infty} g(u,v)du = \int_0^1 ve^{-v}du = ve^{-v}, v \ge 0$$

$$g_U(u)g_V(v) = ve^{-v}, 0 \le u \le 1, v \ge 0 = g(u,v)$$

$\therefore$ U and V are independent.

(ii) Find the correlation coefficient for the following data:

X	10	14	18	22	26	30
Y	18	12	24	6	30	36

Solution:

X	Y	X^2	Y^2	XY
10	18	100	324	180
14	12	196	144	168
18	24	324	576	432
22	6	484	36	132
26	30	676	900	780
30	36	900	1296	1080
120	126	2680	3276	2772

$$n = 6, \sum X = 120, \sum Y = 126, \sum X^2 = 2680, \sum Y^2 = 3276, \sum XY = 2772$$

$$r_{xy} = \frac{n\sum XY - \sum X \sum Y}{\sqrt{n\sum X^2 - (\sum X)^2}\sqrt{n\sum Y^2 - (\sum Y)^2}}$$

$$= \frac{6(2772) - (120)(126)}{\sqrt{6(2680) - (120)^2}\sqrt{6(3276) - (126)^2}} = \frac{1512}{(40.99)(61.48)}$$

$$r_{xy} = 0.6$$

13. (a) (i) Define Poisson process and derive the Poisson probability law.

Solution:

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If $X(t)$represents the number of occurrences of a certain event in $(0, t)$ then the discrete random process $X(t)$ is called the Poisson process, provided the following postulates are satisfied.

(i) P[1 occurrence in $(t, t+\Delta t)] = \lambda\Delta t + O(\Delta t)$

(ii) P[0 occurrence in $(t, t+\Delta t)] = 1 - \lambda\Delta t + O(\Delta t)$

(iii) P[2 or more occurrences in $(t, t+\Delta t)] = O(\Delta t)$

(iv) $X(t)$ is independent of the number of occurrences of the event in any interval prior and after the interval $(0, t)$.

(v) The probability that the event occurs as specified number of times in $(t_0, t_0 + t)$ depends only on t, but not on t_0.

Probability law for the Poisson process $X(t)$:

Let λ be the number of occurrences of the event in unit time.

Let $P_n(t) = P[X(t) = n]$ =probability that there are n occurrences in $(0, t)$

$$P_n(t+\Delta t) = P\big((n-1)\ occurences\ in\ (0,t)\ and\ 1\ occurrences\ in\ (t, t+\Delta t)\big)$$
$$+P\big(n\ occurences\ in\ (0,t)\ and\ 0\ occurrences\ in\ (t, t+\Delta t)\big)$$
$$P_n(t+\Delta t) = P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t)$$
$$P_n(t+\Delta t) = P_{n-1}(t)\lambda\Delta t + P_n(t) - P_n(t)\lambda\Delta t$$
$$P_n(t+\Delta t) - P_n(t) = P_{n-1}(t)\lambda\Delta t - P_n(t)\lambda\Delta t$$
$$\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = P_{n-1}(t)\lambda - P_n(t)\lambda$$

Taking limit as $\Delta t \to 0$, we get

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = P_{n-1}(t)\lambda - P_n(t)\lambda$$
$$\frac{d}{dt}P_n(t) = \lambda[P_{n-1}(t) - P_n(t)] \dots (1)$$

Let the solution of (1) be

$$P_n(t) = \frac{(\lambda t)^n}{n!} f(t) \dots (2)$$

Differentiating (2) with respect to t,

$$\frac{d}{dt}P_n(t) = \frac{\lambda^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] \dots (3)$$

Using (2) and (3) in (1), we get

$$\frac{\lambda^n}{n!}[nt^{n-1}f(t) + t^n f'(t)] = \lambda\frac{(\lambda t)^{n-1}}{(n-1)!}f(t) - \lambda\frac{(\lambda t)^n}{n!}f(t)$$

$$\frac{\lambda^n}{n!}t^n f'(t) = -\lambda\frac{(\lambda t)^n}{n!}f(t) \Rightarrow f'(t) = -\lambda f(t) \Rightarrow f'(t) + \lambda f(t) = 0$$

$$f(t) = ke^{-\lambda t} \ldots (4)$$

From (2) $f(0) =$ probability of number of occurrence in $(0,0) = P_0(0) = 1$

$$f(0) = ke^{-\lambda 0} \Rightarrow k = 1$$

$$f(t) = e^{-\lambda t} \ldots (5)$$

Substituting (5) in (2), we get

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,\ldots$$

(ii) A man goes to his office by car or catches the train every day. He never goes 2 days in a row by train but if he drives one day, then the next day he is just as likely to go by car again as he is to travel by train. Now suppose that on the first day of the week, the man tossed a fair dice and went by car to work if and only if a 6 appeared. Find (1) probability that he went by train on the third day and (2) the probability he went by car to work in a long run.

Solution:

State space is $\{Train\ T,\ Car\ C\}$

The tpm is

$$P = \begin{array}{c} \\ T \\ C \end{array} \begin{array}{c} \begin{array}{cc} T & C \end{array} \\ \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}$$

$$P(going\ by\ car) = P(getting\ six\ in\ the\ toss\ of\ the\ dice) = \frac{1}{6}$$

$$P(going\ by\ train) = 1 - \frac{1}{6} = \frac{5}{6}$$

The initial state probability distribution is

$$P^{(1)} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}$$

$$(1) P^{(2)} = P^{(1)}P = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}$$

$$P^{(3)} = P^{(2)}P = \begin{pmatrix} \frac{1}{12} & \frac{11}{12} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{11}{24} & \frac{13}{24} \end{pmatrix}$$

$$Probability\ that\ he\ went\ by\ train\ on\ the\ third\ day\ = \frac{11}{24}$$

(2) let $\pi = (\pi_1, \pi_2)$ be the stationary state distribution of the Markov chain.

We know that $\pi P = \pi$ and $\pi_1 + \pi_2 = 1 \ldots (1)$

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$$(\pi_1, \pi_2)\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = (\pi_1, \pi_2)$$

$$0\,\pi_1 + \frac{1}{2}\pi_2 = \pi_1 \Rightarrow \pi_2 = 2\pi_1 \ldots (2)$$

$$1\,\pi_1 + \frac{1}{2}\pi_2 = \pi_2 \Rightarrow \pi_2 = 2\pi_1 \ldots (3)$$

Substituting equation (2) or (3) in (1), we get

$$\pi_1 + \pi_2 = 1 \Rightarrow \pi_1 + 2\pi_1 = 1 \Rightarrow 3\pi_1 = 1 \Rightarrow \pi_1 = \frac{1}{3}$$

$$from\ (2), we\ get\ \pi_2 = \frac{2}{3}$$

$$The\ probability\ he\ went\ by\ car\ to\ work\ in\ a\ long\ run\ = \frac{2}{3}$$

(b) (i) Show that the random process $X(t) = A\cos(\omega_0 t + \theta)$ is a wide -sense stationary if A and ω_0 are constants and θ is uniformly distributed random variable in $(0, 2\pi)$.

Solution:

Given θ is uniformly distributed random variable in $(0, 2\pi)$.

$$f(\theta) = \frac{1}{2\pi}, (0 < \theta < 2\pi)$$

$$E[X(t)] = E[A\cos(\omega_0 t + \theta)]$$

$$= A\,E[\cos(\omega_0 t + \theta)] = A\int_0^{2\pi} \cos(\omega_0 t + \theta)\, f(\theta) d\theta$$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega_0 t + \theta)\, d\theta = \frac{A}{2\pi}[\sin(\omega_0 t + \theta)]_0^{2\pi} = \frac{A}{2\pi}[\sin(\omega_0 t + 2\pi) - \sin(\omega_0 t + 0)]$$

$$E[X(t)] = \frac{A}{2\pi}[\sin\omega_0 t - \sin\omega_0 t] = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E[A\cos(\omega_0 t + \theta)\, A\cos(\omega_0(t+\tau) + \theta)]$$

$$= A^2 E\left[\frac{\cos(\omega_0 t + \theta + \omega_0(t+\tau) + \theta) + \cos(\omega_0 t + \theta - \omega_0(t+\tau) - \theta)}{2}\right]$$

$$= \frac{A^2}{2}E[\cos(2\omega_0 t + \omega_0\tau + 2\theta)] + \frac{A^2}{2}E[\cos(\omega_0\tau)]$$

$$= 0 + \frac{A^2}{2}\cos\omega_0\tau$$

$$R_{XX}(t, t+\tau) = \frac{A^2}{2}\cos\omega_0\tau$$

$\therefore$ Mean of $X(t)$ is a constant and auto correlation function of $X(t)$ is dependent on τonly.

$\therefore X(t) = A\cos(\omega_0 t + \theta)$ is a wide -sense stationary

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(ii) If customers arrive at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between 2 consecutive arrivals is (1) more than 1 min. (2) between 1 min and 2 min and (3) 4 min (or) less.

Solution:

We know the property of Poisson process that the inter arrival time T follows exponential distribution with parameter λ.

Here $\lambda = 2$.

$$The\ pdf\ is\ f(x) = \lambda e^{-\lambda t} = 2e^{-2t}.$$

P [interval between 2 consecutive arrivals is more than 1 min] is

$$P[T > 1] = \int_1^\infty f(t)dt = \int_1^\infty 2e^{-2t}dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^\infty = e^{-2}$$

P [between 1 min and 2 min] is

$$P[1 < T < 2] = \int_1^2 f(t)dt = \int_1^2 2e^{-2t}dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^2 = e^{-2} - e^{-4}$$

P [4 min (or) less] is

$$P[T \le 4] = \int_0^4 f(t)dt = \int_0^4 2e^{-2t}dt = 2\left[\frac{e^{-2t}}{-2}\right]_0^4 = 1 - e^{-8}$$

14. (a) Find the mean number of customers in the queue, system, average waiting time in the queue and system of $(M/M/C)$ queueing model.

Solution:

Let $\lambda_n = \lambda$ for all n.

For the Poisson queue system

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n}} \quad \dots (1)$$

$$P_n = \frac{\lambda_0 \lambda_1 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} P_0, n \ge 1 \quad \dots (2)$$

If there is a single server, $\mu_n = \mu$ for all n. But there are C servers working independently of each other. If there be less than C customer. i.e., if $n < C$, only n of the C servers will be busy and the others idle and hence the mean service rate will be null.

If $n \ge C$ all the C serves will be busy and hence the mean source rate $= C\mu$

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$$\therefore \mu_n = \begin{cases} n\mu, & 0 \le n < C \\ C\mu, & n \ge C \end{cases} \quad \dots (3)$$

Using (3) in (1) and (2) we get,

$$P_n = \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0, \quad 0 \le n < C \dots (4)$$

$$P_n = \frac{\lambda^n P_0}{1.2\mu.3\mu \dots (C-1)\mu.(C\mu)(C\mu)(C\mu)\dots(n-C+1)times}, n \ge C$$

$$P_n = \frac{\lambda^n P_0}{(c-1)!\,\mu^{C-1}(C\mu)^{(n-C+1)}}, n \ge c$$

$$P_n = \frac{1}{C!\,(C)^{(n-C)}}\left(\frac{\lambda}{\mu}\right)^n P_0, n \ge c \dots (6)$$

Now P_0 is given by

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=C}^{\infty} \frac{1}{C!\,(C)^{(n-C)}}\left(\frac{\lambda}{\mu}\right)^n\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \sum_{n=C}^{\infty} \frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^n\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{(C)^C}{C!}\left(\frac{\lambda}{\mu C}\right)^C \left(\frac{1}{1-\frac{\lambda}{\mu C}}\right)\right] P_0 = 1$$

$$\left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!\left(1-\frac{\lambda}{\mu C}\right)}\left(\frac{\lambda}{\mu}\right)^C\right] P_0 = 1$$

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{1}{C!\left(1-\frac{\lambda}{\mu C}\right)}\left(\frac{\lambda}{\mu}\right)^C\right]^{-1} \quad \dots (6)$$

$$L_q = E[n-C] = \sum_{n=C}^{\infty} (n-C) P_n$$

$$= \sum_{n=C}^{\infty} (n-C) \frac{1}{C!\,(C)^{(n-C)}}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{(C)^C}{C!} P_0 \sum_{n=C}^{\infty} (n-C)\left(\frac{\lambda}{\mu C}\right)^n$$

$$= \frac{(C)^C}{C!} P_0 \left[\left(\frac{\lambda}{\mu C}\right)^{C+1} + 2\left(\frac{\lambda}{\mu C}\right)^{C+2} + 3\left(\frac{\lambda}{\mu C}\right)^{C+3} + \cdots \right]$$

$$= \frac{(C)^C}{C!} \left(\frac{\lambda}{\mu C}\right)^{C+1} P_0 \left[1 + 2\left(\frac{\lambda}{\mu C}\right)^{1} + 3\left(\frac{\lambda}{\mu C}\right)^{2} + \cdots \right]$$

$$= \frac{(C)^C}{C!} \left(\frac{\lambda}{\mu C}\right)^{C+1} \frac{1}{\left(1 - \frac{\lambda}{\mu C}\right)^2} P_0$$

$$L_s = L_q + \frac{\lambda}{\mu} = \frac{(C)^C}{C!} \left(\frac{\lambda}{\mu C}\right)^{C+1} \frac{1}{\left(1 - \frac{\lambda}{\mu C}\right)^2} P_0 + \frac{\lambda}{\mu}$$

$$W_s = \frac{L_s}{\lambda} = \frac{(C)^C}{\lambda\, C!} \left(\frac{\lambda}{\mu C}\right)^{C+1} \frac{1}{\left(1 - \frac{\lambda}{\mu C}\right)^2} P_0 + \frac{1}{\mu}$$

$$W_q = \frac{L_q}{\lambda} = \frac{(C)^C}{\lambda\, C!} \left(\frac{\lambda}{\mu C}\right)^{C+1} \frac{1}{\left(1 - \frac{\lambda}{\mu C}\right)^2} P_0$$

(b) There are 3 typists in an office. Each typist can type an average of 6 letters per hour. If the letters arrive for being typed at the rate of 15 letters per hour.

(i) What fraction of the time all typists will be busy?

(ii) What is the average number of letters waiting to be typed?

(iii) What is the average time a letter has to spend for waiting and for being typed?

(iv) What is the probability that a letter will take longer than 20 minutes waiting to be typed and being typed?

Solution:

This is of type $(M/M/C):(\infty/FCFS)$

$$C = 3, \lambda = 15 \ letters\ per\ hour$$

$$\mu = 6 \ letters\ per\ minute$$

$$P(All\ typists\ are\ busy) = P(N > C) = \frac{\left(\frac{\lambda}{\mu}\right)^C P_0}{C!\left(1 - \frac{\lambda}{\mu c}\right)}$$

$$P_0 = \left[\sum_{n=0}^{C-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n + \frac{\left(\frac{\lambda}{\mu}\right)^C}{C!\left(1 - \frac{\lambda}{\mu c}\right)} \right]^{-1}$$

$$P_0 = \left[1 + \frac{15}{6} + \frac{1}{2}\left(\frac{15}{6}\right)^2 + \frac{\left(\frac{15}{6}\right)^3}{3!\left(1 - \frac{15}{3(6)}\right)}\right]^{-1} = (22.25)^{-1} = 0.0449$$

$$P(N > C) = \frac{\left(\frac{15}{6}\right)^3}{3!\left(1 - \frac{15}{3(6)}\right)}(0.0449) = 0.7016$$

The fraction of time all the typists will be busy is 0.7016

$$L_q = \frac{1}{C.C!}\frac{\left(\frac{\lambda}{\mu}\right)^{C+1} P_0}{\left(1 - \frac{\lambda}{\mu c}\right)^2}$$

$$= \frac{1}{3.3!}\frac{\left(\frac{15}{6}\right)^{3+1} (0.0449)}{\left(1 - \frac{15}{3(6)}\right)^2} = 3.5078$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\lambda}\left[L_q + \frac{\lambda}{\mu}\right] = \frac{1}{15}\left[3.5078 + \frac{15}{6}\right] = 0.4005\ hours$$

The probability that a letter will take longer than 20 minutes waiting to be typed and being typed is

$$P(W > t) = e^{-\mu t}\left\{1 + \frac{\left(\frac{\lambda}{\mu}\right)^c \left[1 - e^{-\mu t\left(c-1-\frac{\lambda}{\mu}\right)}\right]}{c!\left(1 - \frac{\lambda}{\mu c}\right)\left(c - 1 - \frac{\lambda}{\mu c}\right)} P_0\right\}$$

$$t = 20\ minutes = \frac{1}{3}\ hours$$

$$P\left(W > \frac{1}{3}\right) = e^{-\frac{6}{3}}\left\{1 + \frac{\left(\frac{15}{6}\right)^3 \left[1 - e^{-\frac{6}{3}\left(3-1-\frac{15}{6}\right)}\right]}{3!\left(1 - \frac{15}{6(3)}\right)\left(3 - 1 - \frac{15}{6(3)}\right)} P_0\right\}$$

$$P\left(W > \frac{1}{3}\right) = 0.4616$$

15. (a) Derive the expected steady state system size for the single server queues with Poisson input and General service or Derive Pollaczek-Khintchine formula.

Solution:

Let n and n_1 be the number of customer in the system at times t and $t + T$, when two consecutive customers have just left the system after getting service.

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Let $f(t), E(T)$ and $var(T)$ be the probability density function, mean and variance of T. Let k be the number of customers arriving in the system during the service time T.

Hence

$$n_1 = \begin{cases} k, & if\ n = 0 \\ n - 1 + k, & if\ n > 0 \end{cases}$$

Where $k = 0,1,2,3, \ldots$, is the number of arrivals during the service time. If

$$\delta = \begin{cases} 1\ if\ n = 0 \\ 0\ if\ n > 1 \end{cases}$$

Then $n_1 = n - 1 + \delta + k$ $\ldots(1)$

$$E(n_1) = E(n - 1 + \delta + k) \Rightarrow E(n_1) = E(n) - 1 + E(\delta) + E(k) \quad \ldots(2)$$

When the system has reached the steady state, the probability of the number of customers in the system will be constant. Hence

$$E(n_1) = E(n)\ and\ E(n_1^2) = E(n^2) \quad \ldots(3)$$

Substituting (3) in (2), we get

$$-1 + E(\delta) + E(k) = 0 \Rightarrow E(\delta) = 1 - E(k) \quad \ldots(4)$$

$$n_1^2 = (n + k - 1 + \delta)^2$$

$$n_1^2 = n^2 + (k-1)^2 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta(k-1)$$

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta^2 + 2n(k-1) + 2n\delta + 2\delta k - 2\delta \ldots(5)$$

Since $\delta = \delta^2$ and n $\delta = 0$,we get

$$n_1^2 = n^2 + k^2 - 2k + 1 + \delta + 2n(k-1) + 2\delta k - 2\delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + 2\delta k - \delta$$

$$2n(1-k) = n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)$$

$$E\big(2n(1-k)\big) = E\big(n^2 - n_1^2 + k^2 - 2k + 1 + \delta(2k-1)\big)$$

$$2E(n)E(1-k) = E(n^2) - E(n_1^2) + E(k^2) - 2E(k) + 1 + E(\delta)E(2k-1)$$

$$2E(n)\big(1 - E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1 - E(k)\big)E(2k-1) \quad \text{(Using (3)\& (4))}$$

$$2E(n)\big(1 - E(k)\big) = E(k^2) - 2E(k) + 1 + \big(1 - E(k)\big)(2E(k) - 1)$$

$$E(n) = \frac{\big(E(k^2) - 2E(k) + 1 + \big(1 - E(k)\big)(2E(k) - 1)\big)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-1 + 1 - E(k)\big)(2E(k) - 1)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) + \big(-E(k)\big)(2E(k) - 1)}{2\big(1 - E(k)\big)} = \frac{E(k^2) + E(k) - 2E^2(k)}{2\big(1 - E(k)\big)}$$

$$E(n) = \frac{E(k^2) - E^2(k) + E(k) - E^2(k)}{2(1 - E(k))} = \frac{Var\,(k) + E(k) - E^2(k)}{2(1 - E(k))} \ldots (6)$$

Since the number k of arrivals follows Poisson process with parameter λ.

$$E(k/T) = \lambda T$$

$$E(k^2/T) = \lambda^2 T^2 + \lambda T$$

$$E(k) = \int_0^\infty E(k/T)\, f(t)dt = \lambda \int_0^\infty T\, f(t)dt = \lambda E(T) \ldots (7)$$

$$E(k^2) = \int_0^\infty E(k^2/T)\, f(t)dt = \int_0^\infty (\lambda^2 T^2 + \lambda T)\, f(t)dt = \lambda^2 \int_0^\infty T^2\, f(t)dt + \lambda \int_0^\infty T\, f(t)dt$$

$$E(k^2) = \lambda^2 E(T^2) + \lambda E(T) \ldots (8)$$

$$Var(k) = E(k^2) - E^2(k) = \lambda^2 E(T^2) + \lambda E(T) - \lambda^2 E^2(T) = \lambda^2 \left(E(T^2) - E^2(T)\right) + \lambda E(T)$$

$$Var(k) = \lambda^2 Var(T) + \lambda E(T) \ldots (9)$$

Substituting (7),(8) and (9) in (6), we get

$$E(n) = \frac{\lambda^2 Var(T) + \lambda E(T) + \lambda E(T) - \lambda^2 E^2(T)}{2(1 - \lambda E(T))}$$

$$E(n) = \frac{\lambda^2 Var(T) + \lambda^2 E^2(T) + 2\lambda E(T) - 2\lambda^2 E^2(T)}{2(1 - \lambda E(T))}$$

$$E(n) = \frac{\lambda^2 \left(Var(T) + E^2(T)\right) + 2\lambda E(T)(1 - \lambda E(T))}{2(1 - \lambda E(T))}$$

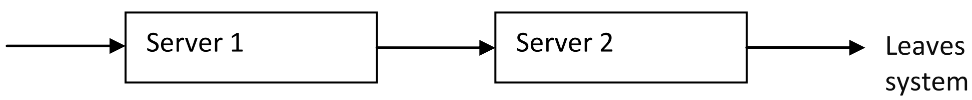
$$E(n) = \lambda E(T) + \frac{\lambda^2 \left(Var(T) + E^2(T)\right)}{2(1 - \lambda E(T))}$$

15. (b) Write Short notes on :

Open and Closed Queue networks

Ans:

Open networks : Consider a two server system in which customers arrive at a Poisson rate λ at server 1. After being served by server 1, they join the queue in front of server 2. We suppose there is infinite waiting space at both servers. Each server serves only one customer at a time with server I taking exponential time with rate μ_i such a system is called a sequential system.



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Consider a system of k servers customers arrive from outside the system to server i. $i = 1,2,\dots,k$ in accordance with independent Poisson process with rate r_i, they join the queue at i until their turn comes one a customer is served by server i, he then joins the queue in front of the server j, with probability P_{ij} or leaves the system with probability P_{i0}. Hence if we let λ_j be the total arrival rate of customers to server j, then

$$\lambda_j = r_j + \sum_{i=1}^{k} \lambda_i P_{ij}$$

which is called flow balance equation.

Closed networks: A queueing network of k nodes is called a closed Jackson network if new customers never enter into and the existing customer never depart from the system i.e $r_i = 0$ and $P_{i0} = 0$ for all i.

The flow balance equation of this model becomes

$$\lambda_j = \sum_{i=1}^{k} \lambda_i P_{ij}$$

Jackson's open network concept can be extended when the nodes are multi server nodes. In this case the network behaves as if each node is an independent $M/M/S$ model.

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