Fourier series

Part - A

1.Define Dirichlet's conditions

Ans:

A function defined in $c \leq x \leq c + 2l$ can be expanded as an infinite trigonometric series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right), \text{provided}$$

i) $f(x)$ is single valued and periodic in $(c, c + 2l)$

ii) $f(x)$ is continuous or piecewise continuous with finite number of finite discontinuous in $(c, c + 2l)$

iii) $f(x)$ has no or finite number of maxima or minima in $(c, c + 2l)$.

2. Write the Fourier series for the function f(x) in the interval $\boldsymbol{c \leq x \leq c + 2l}$***.***

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

Where

$$a_0 = \frac{1}{l} \int_c^{c+2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_c^{c+2l} f(x) cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_c^{c+2l} f(x) sin \frac{n\pi x}{l} dx$$

3. Write the Fourier series for the function f(x) in the interval $\boldsymbol{0 \leq x \leq 2l}$***.***

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

Where

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) sin \frac{n\pi x}{l} dx$$

4. Write the Fourier series for the function f(x) in the interval $-l \leq x \leq l$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right)$$

Where

$$a_0 = \frac{1}{l} \int_{-l}^{l} f(x) dx$$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) sin \frac{n\pi x}{l} dx$$

5. Write the Fourier series for the function f(x) in the interval $0 \leq x \leq 2\pi$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) cos\ nx dx$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) sin\ nx dx$$

6.Write the Fourier series for the function f(x) in the interval $-\pi \leq x \leq \pi$.

Solution:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos\ nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin\ nx dx$$

7. Write the Fourier series for the even function f(x) in the interval $-\pi \leq x \leq \pi$.

(or) Explain half range cosine series in the interval $0 \leq x \leq \pi$.

Solution:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) cos\ nx dx$$

$b_n = 0$, Then the Fourier series is reduced to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

8. Write the Fourier series for the odd function f(x) in the interval $-\pi \le x \le \pi$.

(or) Explain half range sine series in the interval $0 \le x \le \pi$.

Solution:

$$a_0 = 0, \;\; a_n = 0, b_n = \frac{2}{\pi}\int_0^{\pi} f(x) sin\, nxdx,$$

Then the Fourier series is reduced to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

9. Write the Fourier series for the even function f(x) in the interval $-l \le x \le l$.

(or) Explain half range Fourier cosine series in the interval $0 \le x \le l$

Solution:

$$a_0 = \frac{2}{l}\int_0^{l} f(x)dx$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)cos\, \frac{n\pi x}{l} dx$$

$b_n = 0$, Then the Fourier series is reduced to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

10. Write the Fourier series for the odd function f(x) in the interval $-l \le x \le l$.

(or) Explain half range Fourier sine series in the interval $0 \le x \le l$

Solution:

$$a_0 = 0, \;\; a_n = 0, b_n = \frac{2}{l}\int_0^l f(x) sin\, \frac{n\pi x}{l} dx$$

Then the Fourier series is reduced to

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

11. Define periodic function and give examples.

Ans:

A function $f(x)$ is said to be periodic, if and only if $f(x+T) = f(x)$ where T is called period for the function $f(x)$.

Eg: $\sin x \; and \; \cos x$ are periodic functions with period 2π.

12. Write any two advantages of Fourier series.

Ans:

i) Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist.

ii) The Fourier series is useful in expanding the periodic functions. Since outside the closed interval, there exists a periodic extension of the function

iii) Fourier series of a discontinuous function is not uniformly convergent at all points.

iv) Expansion of an oscillation function by Fourier series gives all modes of oscillation which is extremely useful in physics.

13. Define Parseval's identity for the Fourier series in the interval $(\mathbf{0}, \mathbf{2\pi})$

Solution:

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

14. Define Parseval's identity for the Fourier series in the interval $(\mathbf{0}, \mathbf{2l})$

Solution:

$$\frac{1}{l}\int_0^{2l} [f(x)]^2\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

15. Define Parseval's identity for the Fourier series in the interval $(-\pi, \pi)$

Solution:

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

16. Define Parseval's identity for the Fourier series in the interval $(-l, l)$

Solution:

$$\frac{1}{\pi}\int_{-l}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2]$$

17. Define Parseval's identity for the half range Fourier cosine series in the interval $(0, \pi)$

Solution:

$$\frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2]$$

18. Define Parseval's identity for the half range Fourier sine series in the interval $(0, \pi)$

Solution:

$$\frac{2}{\pi}\int_{0}^{\pi}[f(x)]^2\,dx = \sum_{n=1}^{\infty}[b_n^2]$$

19. Define Parseval's identity for the half range Fourier cosine series in the interval $(0, l)$

Solution:

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2]$$

20. Define Parseval's identity for the half range Fourier sine series in the interval $(0, 2l)$

Solution:

$$\frac{2}{l}\int_{0}^{l}[f(x)]^2\,dx = \sum_{n=1}^{\infty}[b_n^2]$$

21. Define Root-Mean Square value of a function $f(x)$

Solution:

RMS value of a function $f(x) = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$

22. Find the value of a_n in the cosine series expansion of $f(x) = k$ in the interval $(0, 10)$.

Solution:

$$a_n = \frac{2}{l}\int_0^l f(x) cos \frac{n\pi x}{l} dx$$

$$= \frac{2}{10}\int_0^{10} k cos \frac{n\pi x}{10} dx$$

$$= \frac{k}{5}\int_0^{10} cos \frac{n\pi x}{10} dx$$

$$= \frac{k}{5}\left[\frac{sin \frac{n\pi x}{10}}{\frac{n\pi}{10}}\right]_0^{10}$$

$$= \frac{k}{5} \times \frac{10}{n\pi}\left[sin \frac{10n\pi}{10} - \sin 0\right]$$

$$a_n = 0$$

23. Find the root mean square value of the function $f(x) = x$ in the interval $(0, l)$

Solution:

RMS value of a function

$$f(x) = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

$$= \sqrt{\frac{\int_0^l x^2 dx}{l-0}}$$

$$= \sqrt{\frac{\left(\frac{x^3}{3}\right)_0^l}{l}}$$

$$= \sqrt{\frac{\frac{l^3}{3}}{l}}$$

$$= \frac{l}{\sqrt{3}}$$

24. State the convergence theorem on Fourier series.

Ans:

i) The Fourier series of $f(x)$ converges to $f(x)$ at all points where $f(x)$ is continuous. Thus, if $f(x)$ is continuous at $x = x_0$, the sum of the Fourier series when $x = x_0$ is $f(x_0)$.

ii) At a point $x = x_0$ where $f(x)$ has a finite discontinuity, the sum of the Fourier series $= \frac{1}{2}[y_1 + y_2]$ where y_1 and y_2 are the two values of $f(x)$ at the point of finite discontinuity.

25. The function $f(x) = \frac{1}{x-2}(0 \le x \le 3)$ cannot be expanded as Fourier series. Explain why?

Ans:

Given $f(x) = \frac{1}{x-2}(0 \le x \le 3)$

At $x = 2, f(x)$ becomes infinity. So it does not satisfy one of the Direchlet's condition. Hence it cannot be expanded as a Fourier series.

26. Can you expand $f(x) = \frac{1-x^2}{1+x^2}$ as a Fourier series in any interval.

Solution:

$$\text{Let } f(x) = \frac{1 - x^2}{1 + x^2}$$

This function is well defined in any finite interval in the range $(-\infty, \infty)$ it has no discontinuous in the interval

$$\text{Differentiate } f'(x) = \frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}$$

$$f'(x) = \frac{-4x}{(1+x^2)^2}$$

$f(x)$ is maximum or minimum when $f'(x) = 0$

(i.e.,) when $4x = 0$ $(i.e.,)$ $x = 0$.

So it has only one extreme value

(i.e.,) a finite number of maxima and minima in the interval $(-\infty, \infty)$

Since it satisfies all the Dirichelt's conditions, it can be expanded in a Fourier series i aspecified interval in the range $(-\infty, \infty)$

27. Find a_0, $f(x) = xsinx$ in $(-\pi < x < \pi)$.
Solution:
Since is $f(x) = xsinx$ is an even function

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\,sinx\,dx$$

$$= \frac{2}{\pi}[x(-\cos x) - 1(-\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}[\pi(-\cos\pi) - 1(-\sin\pi) - 0]$$

$$= 2$$

29. Find a_0, $f(x) = \pi - x^2$ in$(0,\pi)$.
Solution:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi - x^2)dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\pi^2 - \frac{\pi^3}{3}\right)$$

30. Find b_n, $f(x) = x^3$ in $(-\pi, \pi)$.

Solution:

Since $f(x) = x^3$ is an odd function

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^3 sin\, nxdx$$

$$b_n = \frac{2}{\pi}\left[x^3\left(\frac{-cos\, nx}{n}\right) - 3x^2\left(\frac{-sin\, nx}{n^2}\right) + 6x\left(\frac{cos\, nx}{n^3}\right) - 1\left(\frac{sin\, nx}{n^4}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[\pi^3\left(\frac{-cos\, n\pi}{n}\right) - 3\pi^2\left(\frac{-sin\, n\pi}{n^2}\right) + 6\pi\left(\frac{cos\, n\pi}{n^3}\right) - 1\left(\frac{sin\, n\pi}{n^4}\right) - 0\right]$$

$$= \frac{2}{\pi}\left[\pi^3\left(\frac{(-1)^n}{n}\right) + 6\pi\left(\frac{(-1)^n}{n^3}\right)\right]$$

31. Find a_n in $f(x) = e^{-x}$ in $(-\pi, \pi)$.

Solution:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos\, nxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-x}cos\, nxdx$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + nsinnx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{e^{-\pi}}{1+n^2}(\cos n\pi + nsinn\pi) + \frac{e^{\pi}}{1+n^2}(\cos(-n\pi) + nsin(-n\pi))\right]$$

$$= \frac{1}{\pi}\left[-\frac{e^{-\pi}}{1+n^2}(-1)^n + \frac{e^{\pi}}{1+n^2}(-1)^n\right]$$

$$= \frac{(-1)^n}{\pi(1+n^2)}[e^{\pi} - e^{-\pi}]$$

$$= \frac{2(-1)^n}{\pi(1+n^2)} sinh\, x$$

32. Find b_n, $f(x) = \frac{(\pi - x)}{2}$ in $(0, 2\pi)$.

Solution:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) sin\ nx dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)}{2} sin\ nx dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)}{2}\left(-\frac{cos\ nx}{n}\right) - (-1)\left(-\frac{sin\ nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(\pi - 2\pi)}{2}\left(-\frac{cos\ 2n\pi}{n}\right) - (-1)\left(-\frac{sin\ 2n\pi}{n}\right) - \frac{(\pi - 0)}{2}\left(-\frac{cos\ 0}{n}\right) + 0\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2}\left(\frac{1}{n}\right) - 0 - \frac{\pi}{2}\left(-\frac{1}{n}\right) + 0\right] = \frac{1}{n}$$

Fourier Transform

Part-A

1. State the Fourier integral theorem.

Ans: If is a given function $f(x)$ defined in $(-l, l)$ and satisfies Dirichlet's conditions, then

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)dx\,d\lambda$$

2. State the convolution theorem of the Fourier transform.

Ans: if $F(s)$ and $G(s)$ are the functions of $f(x)$ and $g(x)$ respectively then the Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transform

$$F[(f*g)x] = F(S).G(s)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)e^{isx}\,dx = \left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx\right\}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(x)e^{isx}\,dx\right\}$$

3. Write the Fourier transform pairs.

Ans: $F[f(x)]$ and $F^{-1}[F(s)]$ are Fourier transform pairs.

4. Find the Fourier sine transform of $f(x) = e^{-ax}(a > 0)$.

Ans: The Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f(x)\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_{0}^{\infty} e^{-ax}\sin sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\frac{s}{s^2+a^2}\quad\left[Since\int_{0}^{\infty} e^{-ax}\sin bx\,dx = \frac{b}{b^2+a^2}\right]$$

5. What is the Fourier transform of $f(x-a)$ if the Fourier transform of $f(x)$ is $F(s)$.

Solution: $F\big(f(x-a)\big) = e^{ias}\,F(s)$

6. State the Fourier transforms of the derivatives of a function.

Solution: $F\left(\frac{d^n f(x)}{dx^n}\right) = (-is)^n F(s)$.

7. Find the Fourier sine transform of e^{-x}.

Solution: The Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2+1} \quad \left[Since \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{b^2+a^2} \right]$$

8. Prove that $F_c[f(ax)] = \frac{1}{a} F_c\left(\frac{s}{a}\right), a > 0.$

Solution:

$$F_c[f(ax)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(ax) \cos sx \, dx$$

Put $ax = y$ when $x = 0, y = 0.$

$dx = \frac{dy}{a}$ When $x = \infty, y = \infty$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos\left(\frac{sy}{a}\right) \frac{dy}{a}$$

$$= \frac{1}{a} \sqrt{\frac{2}{\pi}} \int_0^\infty f(y) \cos\left(\frac{s}{a}\right) y \, dy$$

$$= \frac{1}{a} F_c\left(\frac{s}{a}\right)$$

9. Find Fourier sine transform of e^{ax}.

Solution: The Fourier sine transform of $f(x)$ is

The Fourier sine transform of $f(x)$ is

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{s}{s^2 + a^2} \quad \left[Since \int_0^\infty e^{ax} \sin bx \, dx = \frac{b}{b^2 + a^2} \right]$$

10. if $F(s) = F(f(x))$, then prove that $F(xf(x)) = (-i) \frac{d[F(s)]}{ds}$

Solution: $F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$

$$\frac{d[F(s)]}{ds} = \frac{d}{ds} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{\partial}{\partial s} (e^{isx}) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) ix (e^{isx}) dx$$

$$= iF(xf(x))$$

$$F(xf(x)) = (-i) \frac{d[F(s)]}{ds}$$

11. Find Fourier sine transform of $\frac{1}{x}$

$$F_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi} \frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}$$

$$\left[Since \int_0^\infty \frac{1}{x} \sin ax \, dx = \frac{\pi}{2}, a > 0 \right].$$

12. Find the Fourier cosine transform of e^{-x}.

Solution: The Fourier sine transform of $f(x)$ is

$$F_c[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{s^2+1} \quad \left[Since \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{b^2+a^2}\right]$$

13. If $F_s[s]$ is the Fourier sine transform of $f(x)$, show that

$$\boldsymbol{F_s[f(x)\cos ax] = \frac{1}{2}[F_s(s+a) + F_s(s-a)]}$$

Solution:

$$F_s[f(x)cosax\,] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos ax \, \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} . \frac{1}{2} \int_0^\infty f(x) \{\sin(a+s)x - \sin(a-s)\,x\,\} \, dx$$

$$= \sqrt{\frac{2}{\pi}} . \frac{1}{2} \int_0^\infty f(x) \, \sin(a+s)x \, dx - \sqrt{\frac{2}{\pi}} . \frac{1}{2} \int_0^\infty f(x) \, \sin(a-s)x \, dx$$

$$= \frac{1}{2}[F_s(s+a) + F_s(s-a)]$$

14. **If $F(s)$ is the Fourier transform of $f(x)$, write the formula for the Fourier transform of $f(x)cos\ ax$ in terms of F.**

Solution: $F[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cos ax \, e^{isx} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2}\right) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{i(s+a)x}}{2} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \frac{e^{i(s-a)x}}{2} dx$$

$$F[f(x)\cos ax] = \frac{1}{2}[F(s+a) + F(s-a)]$$

15. **Find the Fourier transform of** $\boldsymbol{f(x) = \begin{cases} 1 \ in \ |x| < a \\ 0 \ in \ |x| > a \end{cases}}$

Solution: We know that the Fourier transform fo $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{isx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is} \right)_{-a}^{a} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{ias} - e^{-ias}}{is} \right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin as}{s}$$

16. **State any two properties of Fourier transform**

Solution: i) Linearity property: If $f(x)$ and $g(x)$are any two functions then

$F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)]$, where a and b are constants.

ii)Shifting property: If $F[f(x)] = F(s)$, then $F[f(x-a)] = e^{ias} F(s)$

17. **Define the infinite Fourier cosine transform and its inverse**

Solution: The infinite Fourier cosine transform is given by

$$F_c[f(x)] = F_c[s] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx \, dx$$

Then the inverse Fourier cosine transform is given by

$$F_c^{-1}[F_c[s]] = f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c[s] \cos sx \, ds$$

18. **State Parseval's identity for Fourier transform.**

Solution: If $F(s)$ is the Fourier transform of $f(x)$, then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |F(s)|^2 \, ds$$

19. Define the Fourier sine transform of $f(x)$ in $(0, l)$. Also give the inversion formula

Solution: The finite Fourier sine transform of a function $f(x)$ in $(0, l)$is given by

$$f_s(n) = \int_0^l f(x) sin \frac{n\pi x}{l} dx,$$

Then the inversion formula is given by

$$f(x) = \frac{2}{l} \sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{l}.$$

20. **Define the Fourier cosine transform of $f(x)$ in $(0, l)$. Also give the inversion formula**

Solution: The finite Fourier cosine transform of a function $f(x)$ in $(0, l)$is given by

$$f_c(n) = \int_0^l f(x) c \frac{n\pi x}{l} dx,$$

Then the inversion formula is given by

$$f(x) = \frac{1}{l} f_c(0) + \frac{2}{l} \sum_{n=1}^{\infty} f_c(n) \cos \frac{n\pi x}{l}.$$

21. **Define the infinite Fourier sine transform and its inverse**

Solution: The infinite Fourier sine transform is given by

$$F_s[f(x)] = F_s[s] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx \, dx$$

Then the inverse Fourier cosine transform is given by

$$F_s^{-1}[F_s[s]] = f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s[s] \sin sx \, ds$$

22. **Prove that if $f(x)$ is an even function of x, its Fourier transform $F(s)$ will also be an even function of s.**

Solution: By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx \ \dots \ (1)$$

Changing s into –s in both sides of (1),

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx \ \ldots \ (2)$$

In the right hand side integral in (2), put $x = -u$.

then $dx = -du$; when $x = \infty, u = -\infty$

and when $x = -\infty, u = \infty$.

So (2) becomes

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u) e^{isu} . (-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-u) e^{isu} \, du$$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x) e^{isx} \, dx$ [Changing the dummy variable u into x]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx = F(S) \ by \ (1).$$

Here $F(S)$ is an even function of s.

23. **Prove that if $f(x)$ is an odd function of x, its Fourier transform $F(s)$ will also be an odd function of s.**

Solution: By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx \ \ldots \ (1)$$

Changing s into –s in both sides of (1),

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx \ \ldots \ (2)$$

In the right hand side integral in (2), put $x = -u$.

then $dx = -du$; when $x = \infty, u = -\infty$

and when $x = -\infty, u = \infty$.

So (2) becomes

$$F(-s) = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(-u)e^{isu}.(-du)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-u)e^{isu}\, du$$

$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-x)e^{isx}\, dx$ [Changing the dummy variable u into x]

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} -f(x)e^{isx}\, dx = -F(S)\ by\ (1).$$

Here $F(S)$ is an odd function of s.

24. **Find the Fourier transform of the function defined by**

$$\boldsymbol{f(x)} = \begin{cases} \mathbf{0}, & \boldsymbol{x} < a \\ \mathbf{1}, & \boldsymbol{a} < x < b \\ \mathbf{0}\,, & \boldsymbol{x} > b \end{cases}$$

Solution:

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{a} f(x)e^{isx}\, dx + \int_{a}^{b} f(x)e^{isx}\, dx + \int_{b}^{\infty} f(x)e^{isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{isx}}{is} \right]_{a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{isb} - e^{isa}}{is}$$

25**. Show that $\boldsymbol{f(x) = 1, 0} < x < \infty$ cannot be represented by a Fourier integral.**

Solution: $\int_0^\infty |f(x)| dx = \int_0^\infty dx = [x]_0^\infty = \infty - 0 = \infty$

$$i.e., \int_0^\infty |f(x)|dx \text{ is not convergent}$$

Hence $f(x) = 1$ cannot be represented by a Fourier integral.

26. **Find the Fourier cosine transform of $f(x) = \begin{cases} \mathbf{1}, & 0 < x < a \\ \mathbf{0}, & x > a \end{cases}$**

Solution:

We know that

$$F_c[f(x)] = F_c[s] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 1 \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s}\right]_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s} - 0\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sa}{s}\right]$$

27. **Find the Fourier transform of $f(x) = \begin{cases} \mathbf{1} \ \boldsymbol{in} \ |\boldsymbol{x}| \le \mathbf{1} \\ \mathbf{0} \ \boldsymbol{in} \ |\boldsymbol{x}| > 1 \end{cases}$**

Solution: We know that the Fourier transform fo $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{isx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{e^{isx}}{is}\right)_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \left(\frac{e^{is} - e^{-is}}{is}\right)$$

$$= \sqrt{\frac{2}{\pi}} \frac{\sin s}{s}$$

28. **Find the Fourier transform of $e^{-a|x|}, a > 0$.**

Solution: We know that the Fourier transform fo $f(x)$ is given by

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f(x) e^{isx}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(is+a)x}\, dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{(is-a)x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\left[\frac{e^{(is+a)x}}{is+a} \right]_{-\infty}^{0} + \left[\frac{e^{-(a-is)x}}{is-a} \right]_{0}^{\infty} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{is+a} - \frac{1}{is-a} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{is-a-is-a}{(is)^2-a^2}$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{a}{s^2+a^2} \right)$$

29. **State the Fourier transforms of derivates of a function.**

Solution:

$$F\left[\frac{df(x)}{dx}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\, d[f(x)]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[e^{isx} f(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x) is e^{isx}\, dx \right\}$$

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

(assuming $f(x) \to 0\ as\ x \to \pm\infty$)

$$= -isF(s).$$

$$F\left[\frac{d^2 f(x)}{dx^2}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f''(x)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\,d[f'(x)]$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\left[e^{isx} f'(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)ise^{isx}\,dx\right\}$$

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x)e^{isx}\,dx$$

(assuming $f'(x) \to 0\ as\ x \to \pm\infty$)

$$= -\frac{is}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx}\,d[f(x)]$$

$$= -\frac{is}{\sqrt{2\pi}}\left\{\left[e^{isx} f(x)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)ise^{isx}\,dx\right\}$$

$$= \frac{(-is)^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{isx}\,dx$$

(assuming $f'(x) \to 0\ as\ x \to \pm\infty$)

$$= (-is)^2 F(s).$$

In general,

$$F\left[\frac{d^n f(x)}{dx^n}\right] = (-is)^n F(s).$$

30**. Find the finite Fourier cosine transform of $\boldsymbol{f(x) = e^{-ax}}$in $(\boldsymbol{0}, \boldsymbol{l})$.**

Solution: $f_c(n) = \int_0^l f(x)cos\frac{n\pi x}{l}\,dx$, n being a positive integer

$$= \int_0^l e^{-ax} cos \frac{n\pi x}{l} dx$$

$$= \left[\frac{e^{-ax}}{a^2 + \left(\frac{n\pi x}{l}\right)^2} \left(-acos \frac{n\pi x}{l} + \frac{n\pi x}{l} sin \frac{n\pi x}{l} \right) \right]_0^l$$

$$= \frac{l^2 e^{-al}}{(la)^2 + (n\pi)^2} (-a \cos n\pi) - \frac{l^2}{(la)^2 + (n\pi)^2} (-a)$$

$$= \frac{al^2}{(la)^2 + (n\pi)^2} [1 + e^{-al}(-1)^{n+1}]$$

Partial Differential Equations

Part-A

1**. Form a partial differential equation by eliminating arbitrary constants a and b from**

$$\boldsymbol{z = (x + a)^2 + (y + b)^2}$$

Solution: Given

$$z = (x + a)^2 + (y + b)^2 \ldots (1)$$

$$p = \frac{\partial z}{\partial x} = 2(x + a) \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = 2(y + b) \ldots (3)$$

Substituting (2)& (3) in (1), we get $z = \frac{p^2}{4} + \frac{q^2}{4}$

2. **Solve $(\boldsymbol{D^2 - 2DD' + D'^2})\boldsymbol{z = 0}$**

Solution: A.E is $m^2 - 2m + 1 = 0$

$$(m - 1)^2 = 0$$

$$m = 1, 1$$

$$y = f_1(y + x) + xf_2(y + x)$$

3. **Form a partial differential equation by eliminating arbitrary constants a and b from**

$$\boldsymbol{(x - a)^2 + (y - b)^2 = z^2 cot^2\, \alpha}$$

Solution: Given

$$(x - a)^2 + (y - b)^2 = z^2 cot^2\, \alpha \ldots (1)$$

Partially differentiating w.r.t 'x' and 'y' we get

$$2(x-a) = 2zpcot^2\,\alpha \ldots (2)$$

$$2(y-b) = 2zqcot^2\,\alpha \ldots (3)$$

$$(x-a) = zpcot^2\,\alpha \ldots (4)$$

$$(y-b) = zqcot^2\,\alpha \ldots (5)$$

Substituting (4)& (5) in (1), we get $z^2p^2cot^4\,\alpha + \; z^2q^2cot^4\,\alpha = z^2cot^2\,\alpha$

$$p^2 + q^2 = tan^2\alpha$$

4. **Find the complete solution of the differential equation**

$$\boldsymbol{p^2 + q^2 - 4pq = 0}$$

Solution: Given

$$p^2 + q^2 - 4pq = 0 \ldots (1)$$

Let us assume that $z = ax + by + c \ldots (2)$ be the solution of (1).

Partially differentiating (2) w.r.t 'x' and 'y' we get

$$\left.\begin{aligned} p = \frac{\partial z}{\partial x} = a \\ q = \frac{\partial z}{\partial y} = b \end{aligned}\right\} \ldots (3)$$

Substituting (3) in (1) we get

$$a^2 + b^2 - 4ab = 0 \ldots (4)$$

From (4) we get $a = \frac{4b \pm \sqrt{16b^2 - 4a^2b^2}}{2} = \frac{4b \pm 2b\sqrt{4-a^2}}{2} = b \pm b\sqrt{4-a^2} \ldots (5)$

Substituting (5) in (2) we get

$$z = b \pm b\sqrt{4-a^2}x + by + c$$

5.**Find the solution of $\boldsymbol{px^2 + qy^2 = z^2}$**

Solution: The S.E is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$$

Taking 1st two members, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating we get

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

i.e., $\left(\frac{1}{y} - \frac{1}{x}\right) = c_1$

taking last two members, we get

$$\frac{dy}{y^2} = \frac{dz}{z^2}$$

$$-\frac{1}{y} = -\frac{1}{z} + c_2$$

i.e., $\left(\frac{1}{z} - \frac{1}{y}\right) = c_2$

The complete solution is $\Phi\left[\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{y}\right] = 0$

6**. Find the partial differential equation of all planes having equal intercepts on the x and y axis.**

Solution:

The equation of the plane is

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{b} = 1 \ldots (1)$$

Partially differentiating (1) w.r.t 'x' and 'y' we get

$$\frac{1}{a} + \frac{p}{b} = 0$$

$$p = -\frac{b}{a} \ldots (2)$$

$$\frac{1}{a} + \frac{q}{b} = 0$$

$$q = -\frac{b}{a} \ldots (3)$$

From (2) and (3) we get $p = q$

7. **Form the partial differential equation by eliminating the arbitrary function from**

$$\boldsymbol{\Phi\left[z^2 - xy, \frac{x}{z}\right] = 0}$$

Solution: Given

$$\Phi\left[z^2 - xy, \frac{x}{z}\right] = 0 \ldots (1)$$

Let $u = z^2 - xy$

$$v = \frac{x}{z}$$

Then the given equation is of the form

$$\Phi[u, v] = 0 \ldots (2)$$

The elimination of Φ from the equation (2), we get,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

i.e., $\begin{vmatrix} 2zp - y & \frac{z-px}{z^2} \\ 2zq - x & \frac{-xq}{z^2} \end{vmatrix} = 0$

i.e., $(2zp - y)\left(\frac{-xq}{z^2}\right) - (2zq - x)\left(\frac{z-px}{z^2}\right) = 0$

i.e.,$px^2 - q(xy - 2z^2) = zx$

8. **Find the singular integral of the partial differential equation $\boldsymbol{z = px + qy + p^2 - q^2}$**

Solution: The complete integral is $z = ax + by + a^2 - b^2 \ldots (1)$

Now, $\left.\begin{aligned} \frac{\partial z}{\partial a} &= x + 2a = 0 \Rightarrow a = \frac{-x}{2} \\ \frac{\partial z}{\partial b} &= y - 2b = 0 \Rightarrow b = \frac{y}{2} \end{aligned}\right\} \ldots (2)$

Substituting (2) in (1), we get

$$z = -\frac{x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4} = -\frac{x^2}{4} + \frac{y^2}{4}$$

i.e., $y^2 - x^2 = 4z$ which is the singular integral.

9**. Find the complete solution of $x^2p^2 + y^2q^2 = z^2$**

Solution: Given $x^2p^2 + y^2q^2 = z^2 \dots (1)$

Equation (1) can be written as $(xp)^2 + (yq)^2 = z^2 \dots (2)$

Put $\log x = X$ and $\log y = Y$

$$\left.\begin{aligned} xp = P, where\ P = \frac{\partial z}{\partial X} \\ yq = Q, where\ Q = \frac{\partial z}{\partial Y} \end{aligned}\right\} \dots (3)$$

$$P^2 + Q^2 = z^2 \dots (4)$$

This is of the form $F(z, P, Q) = 0$

Let $z = f(X + aY)$ be the solution of (4).

Put $u = X + aY$.Then $z = f(u)$

$$P = \frac{dz}{du}, \qquad Q = a\frac{dz}{du} \dots (5)$$

Substituting (5) in (4), we get

$$\left(\frac{dz}{du}\right)^2 [1 + a^2] = z^2$$

$$\frac{dz}{du} = \frac{z}{\sqrt{1 + a^2}}$$

Separating the variables we get

$$\frac{dz}{z} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating we get

$$\log z = \frac{u}{\sqrt{1 + a^2}} + b$$

$$\log z = \frac{X + aY}{\sqrt{1 + a^2}} + b$$

Replacing $X\ by \log x\ and\ Y\ by \log y$, we get

$$\log z = \frac{log\ x + alog\ y}{\sqrt{1+a^2}} + b$$

Which gives the complete solution of (1).

10.**Solve $p^2 + q^2 = m^2$**

Solution: $p^2 + q^2 = m^2 \ldots (1)$

Let us assume that $z = ax + by + c$ be a solution of (1). ... (2)

Partially differentiating (2) w.r.t $'x'$ and $'y'$, we get

$$\frac{\partial z}{\partial x} = p = a, \frac{\partial z}{\partial y} = q = b \ldots (3)$$

Substituting (3) in (1) we get

$$a^2 + b^2 = m^2 \ldots (4)$$

Hence $z = ax + by + c$ is the solution of (1).

11. **Form a partial differential equation by eliminating arbitrary constants a and b from**

$$\boldsymbol{z = ax^n + by^n}$$

Solution: Given

$$z = ax^n + by^n \ldots (1)$$

$$p = \frac{\partial z}{\partial x} = a.nx^{n-1} \Rightarrow a = \frac{p}{nx^{n-1}} \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = a.ny^{n-1} \Rightarrow b = \frac{q}{ny^{n-1}} \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$z = \frac{p}{nx^{n-1}} x^n + \frac{q}{ny^{n-1}} y^n$$

$$z = \frac{1}{n}(px + qy)$$

12. **Solve $\left(D^3 + 3D^2D' + 3DD'^2 + D'^3\right)z = 0$.**

Solution:

A.E is $m^3 + 3m^2 + 3m + 1 = 0$

$$(m+1)^3 = 0$$

i.e., $m = -1, -1, -1$.

$$z = f_1(y-x) + xf_2(y-x) + x^2 f_3(y-x)$$

13. **Form a partial differential equation by eliminating arbitrary constants a and b from**

$$\boldsymbol{z = (x^2 + a^2)(y^2 + b^2)}$$

Solution:

Given

$$z = (x^2 + a^2)(y^2 + b^2) \ldots (1)$$

$$p = \frac{\partial z}{\partial x} = 2x(y^2 + b^2) \Rightarrow \frac{p}{2x} = y^2 + b^2 \ldots (2)$$

$$q = \frac{\partial z}{\partial y} = 2y(x^2 + a^2) \Rightarrow \frac{q}{2y} = (x^2 + a^2) \ldots (3)$$

Substituting (2) and (3) in (1) we get

$$z = \frac{p}{2x}\frac{q}{2y}$$

$$pq = 4xyz$$

14. **Solve $\boldsymbol{(D^2 - DD' + D' - 1)z = 0}$.**

Solution: The given equation can be written as

$$(D-1)(D-D'+1)z = 0.$$

We know that the complementary function corresponding to the factors

$$(D - m_1 D' - \alpha_1)(D - m_2 D' - \alpha_2)z = 0 \; is$$

$$z = e^{\alpha_1 x} f_1(y + m_1 x) + e^{\alpha_2 x} f_2(y + m_2 x)$$

Here $\alpha_1 = 1, \alpha_2 = -1, m_1 = 0, m_2 = 1$.

$$z = e^x f_1(y) + e^{-x} f_2(y + x)$$

15. **Form the partial differential equation by eliminating the arbitrary function from**

$$\boldsymbol{z = f\left(\frac{xy}{z}\right)}$$

Solution:

$$z = f\left(\frac{xy}{z}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\frac{zy - xyp}{z^2} \dots (1)$$

$$q = f'\left(\frac{xy}{z}\right)\frac{zx - xyq}{z^2} \dots (2)$$

From (1) we get

$$f'\left(\frac{xy}{z}\right) = \frac{pz^2}{zy - xyp} \dots (3)$$

Substituting (3) in (2), we get

$$q = \frac{pz^2}{zy - xyp}\frac{zx - xyq}{z^2}$$

16. **Form a partial differential equation by eliminating arbitrary constants a and b from**

$$\boldsymbol{(x-a)^2 + (y-b)^2 + z^2 = 1}$$

Solution: Given

$$(x-a)^2 + (y-b)^2 + z^2 = 1 \dots (1)$$

Partially differentiating (1) w.r.t. $'x'$ and $'y'$, we get

$$2(x-a) + 2zp = 0$$

i.e., $x - a = -zp \dots (2)$

$$2(y-b) + 2zq = 0$$

$$y - b = -zq \dots (3)$$

Substituting (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = 1$$

$$p^2 + q^2 + 1 = \frac{1}{z^2}$$

17. **Find the complete integral of $\boldsymbol{p + q = pq}$**

Solution: Given $p + q = pq \dots (1)$

Let us assume that $z = ax + by + c$ be a solution of (1). ... (2)

Partially differentiating (2) w.r.t $'x'$ and $'y'$, we get

$$\frac{\partial z}{\partial x} = p = a, \frac{\partial z}{\partial y} = q = b \dots (3)$$

Substituting (3) in (1) we get

$$a + b = ab \dots (3)$$

$$b = \frac{a}{a-1} \dots (4)$$

Substituting (4) in (2) we get

$$z = ax + \left(\frac{a}{a-1}\right) y + c$$

which is the complete integral.

18. **Solve $\left(D^3 - 3DD'^2 + 2D'^3\right)z = 0$.**

Solution:

A.E. is $m^3 - 3m + 2 = 0$

$$m = 1, 1 \; and - 2.$$

The solution is $z = f_1(y + x) + x f_2(y + x) + f_3(y - 2x)$.

19. **Find the general solution of $4\frac{\partial^2 z}{\partial x^2} - 12\frac{\partial^2 z}{\partial x \partial y} + 9\frac{\partial^2 z}{\partial y^2} = 0$.**

Solution:

A.E. is $4m^2 - 12m + 9 = 0$

$$m = \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$m = \frac{3}{2}, \frac{3}{2}$$

The general solution is $z = f_1\left(y + \frac{3}{2}x\right) + x f_2\left(y + \frac{3}{2}x\right)$

20. **Form the partial differential equation by eliminating the arbitrary function from**

$$\boldsymbol{z = f\left(\frac{y}{x}\right)}$$

Solution:

$$z = f\left(\frac{y}{x}\right) \dots (1)$$

Partially differentiating (1) w.r.t $'x'$ and $'y'$, we get

$$p = f'\left(\frac{y}{x}\right)\frac{-y}{x^2} \dots (2)$$

$$q = f'\left(\frac{y}{x}\right)\frac{1}{x} \dots (3)$$

From (1) we get

$$f'\left(\frac{y}{x}\right) = -\frac{px^2}{y} \dots (4)$$

Substituting (4) in (3), we get

$$q = -\frac{px^2}{y}\frac{1}{x}$$

$px + qy = 0$ is the required partial differential equation.

21. **Form the partial differential equation by eliminating the arbitrary function from**

$$z = x + y + f(xy).$$

Solution: Given $z = x + y + f(xy) \dots (1)$

Partially differentiating (1) w.r.t $'x'$ and $'y'$, we get

$$p = 1 + f'(xy)y \Rightarrow p - 1 = f'(xy)y \dots (2)$$

$$q = 1 + f'(xy)x \Rightarrow q - 1 = f'(xy)x \dots (3)$$

$(2) \div (3)$ gives $\frac{p-1}{q-1} = \frac{y}{x}$

$px - x = qy - y \text{ or } px - qy = x - y$ is the required partial differential equation.

22. **Form the partial differential equation by eliminating the arbitrary function from**

$$z = f(xy).$$

Solution: Given $z = f(xy) \dots (1)$

Partially differentiating (1) w.r.t $'x'$ and $'y'$, we get

$$p = f'(xy)y \dots (2)$$

$$q = f'(xy)x \dots (3)$$

$(2) \div (3)$ gives $\frac{p}{q} = \frac{y}{x}$

$px = qy \text{ or } px - qy = 0$ is the required partial differential equation.

23. **Find the partial differential equation of all planes through the origin.**

Solution:

The general equation of the plane is

$$ax + by + cz + d = 0 \dots (1)$$

If (1) passes through the origin, then d=0.

So (1) becomes

$$ax + by + cz = 0 \dots (2)$$

Partially differentiating (2) w.r.t 'x' and 'y' we get

$$a + cp = 0 \Rightarrow a = -cp \dots (3)$$

$$b + cq = 0 \Rightarrow b = -cq \dots (4)$$

Substituting (3) and (4) in (1), we get

$-cpx - cqy + cz = 0 \Rightarrow px + qy = z$ is the required partial differential equation.

24. **Write down the complete solution of $\boldsymbol{z = px + qy + c\sqrt{1 + p^2 + q^2}}$.**

Solution: The given equation is Clairaut's type.

So replacing p by a and q by b, the complete solution is

$$z = ax + by + c\sqrt{1 + a^2 + b^2}$$

25. **Find the differential equations of all spheres of radius c having their centers in the xoy plane.**

Solution: Let the center of the sphere be $(a, b, 0)$, a point in the xoy plane. c is the given radius.

The equation of the sphere is $(x - a)^2 + (y - b)^2 + z^2 = c^2 \dots (1)$

Partially differentiating (1) w.r.t 'x' and 'y' we get

$$2(x - a) + 2zp = 0 \Rightarrow x - a = -zp \dots (2)$$

$$2(y - b) + 2zq = 0 \Rightarrow y - b = -zq \dots (3)$$

Substituting (2) and (3) in (1), we get

$$z^2p^2 + z^2q^2 + z^2 = c^2$$

$z^2(p^2 + q^2 + 1) = c^2$ which is the required partial differential equation.

26. **Eliminate f from $xyz = f(x + y + z)$**

Solution: Given $xyz = f(x + y + z) \dots (1)$.

Partially differentiating (1) w.r.t 'x' and 'y' we get

$$y(z + xp) = f'(x + y + z)(1 + p) \dots (2)$$

$$x(z + yq) = f'(x + y + z)(1 + q) \dots (3)$$

(2) ÷ (3) gives $\frac{(1+p)}{(1+q)} = \frac{y(z+xp)}{x(z+yq)}$

$$(1 + p)x(z + yq) = (1 + q)y(z + xp)$$

$$p(xy - xz) + q(yz - xy) = xz - yz$$

$px(y - z) + qy(z - x) = z(x - y)$ is the required partial differential equation.

27. Mention three types of partial differential equation.

Solution: i) A solution of a partial differential equation which contains the maximum possible number of arbitrary constants is called complete integral.

ii) A solution obtained by giving particular values to the arbitrary constants in a complete integral is called a particular integral.

iii) A solution of a partial differential equation which contains the maximum possible number of arbitrary functions is called general integral.

28. **Define singular integral.**

Solution:

Let $F(x, y, z, p, q) = 0 \dots (1)$be a partial differential equation and its complete integral be

$$\Phi(x, y, z, a, b) = 0 \dots (2)$$

Differentiating (2) partially w.r.t. a and b in turn, we get

$$\frac{\partial \Phi}{\partial a} = 0 \dots (3)$$

$$\frac{\partial \Phi}{\partial b} = 0 \dots (4)$$

The eliminant of a and b from (2), (3) and (4), if it exits, is called singular integral.

29. **Solve $\sqrt{p}+\sqrt{q}=1$**

Solution: Given $\sqrt{p}+\sqrt{q}=1 \dots (1)$

Let us assume that $z=ax+by+c$ be a solution of (1). ... (2)

Partially differentiating (2) w.r.t $'x'$ and $'y'$, we get

$$\frac{\partial z}{\partial x}=p=a, \frac{\partial z}{\partial y}=q=b \dots (3)$$

Substituting (3) in (1), we get

$$\sqrt{a}+\sqrt{b}=1 \Rightarrow b=\left(1-\sqrt{a}\right)^2 \dots (4)$$

Substituting (4) in (2), we get

$$z=ax+\left(1-\sqrt{a}\right)^2 y+c \text{ is the complete integral of (1).}$$

30. **Find the particular integral of $\left(D^3-3D^2D'-4DD'^2+12D'^3\right)z=\sin(x+2y)$**

Solution:

Particular integral $=\frac{\sin(x+2y)}{\left(D^3-3D^2D'-4DD'^2+12D'^3\right)} \dots (1)$

Replacing $D^2\ by-1^2\ and D'^2 by-2^2$ in(1)

$$=\frac{\sin(x+2y)}{(-D+3D'+16D-48D')}$$

$$=\frac{\sin(x+2y)}{15D-45D'}$$

$$=\frac{\sin(x+2y)}{15(D-3D')}$$

$$=\frac{(D+3D')\sin(x+2y)}{15(D^2-9D'^2)}$$

$$=\frac{\cos(x+2y)+6\cos(x+2y)}{15(D^2-9D'^2)}$$

$$=\frac{7\cos(x+2y)}{15(-1-9(-4))}$$

$$=\frac{\cos(x+2y)}{75}$$

Z –Transform

Part-A

1. Find $Z\{n\}$.

Solution:

$$Z(n) = \sum_{n=0}^{\infty} nz^{-n} = \sum_{n=0}^{\infty} \frac{n}{z^n} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots\right] = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-2} = \frac{1}{z}\frac{z^2}{(z-1)^2}$$

$Z(n) = \frac{z}{(z-1)^2}$ [The region of convergence is $\left|\frac{1}{z}\right| < 1 \; or \; |z| > 1$]

2. Form the difference equation from $y_n = a + b3^n$

Solution: $y_n = a + b3^n, y_{n+1} = a + b3^{n+1`}, y_{n+2} = a + b3^{n+2}$

$$y_{n+2} - 4y_{n+1} + 3y_n = 0$$

3. Find the value of $Z[f(n)]$ when $f(n) = na^n$

Solution: $Z(na^n) = \{Z(n)\}_{z \to \frac{z}{a}}$ $\left[Since\; Z[a^n f(t)] = F\left(\frac{z}{a}\right)\right]$

$$= \left\{\frac{z}{(z-1)^2}\right\}_{z \to \frac{z}{a}} = \frac{\frac{z}{a}}{\left(\frac{z}{a} - 1\right)^2} = \frac{az}{(z-a)^2}.\left[Since\; Z(n) = \frac{z}{(z-1)^2}\right]$$

4. Find $Z\left[\frac{a^n}{n!}\right]$ in Z-transform.

Solution: $Z\left[\frac{a^n}{n!}\right] = \left\{Z\left(\frac{1}{n!}\right)\right\}_{z \to \frac{z}{a}}$ $\left[Since\; Z[a^n f(t)] = F\left(\frac{z}{a}\right)\right]$

$$= \left\{e^{1/z}\right\}_{z \to \frac{z}{a}} = e^{a/z}$$

5.Find $Z[e^{-iat}]$ using Z-transform.

Solution: $Z\left[e^{-iat}\right] = Z\left[e^{-iat}.1\right]$

$$= \{Z(1)\}_{z \to ze^{iaT}} \left[Since\; Z\left[e^{-iat}.f(t)\right] = F(ze^{iaT})\right]$$

$$= \left\{\frac{z}{z-1}\right\}_{z \to ze^{iaT}}$$

$$= \frac{ze^{iaT}}{ze^{iaT} - 1}$$

6. Express $Z[f(n+1)]$ in terms of $F(z)$.

Solution: $Z[f(n+1)] = zF(z) - zf(0)$.

7. State and prove initial value theorem in Z-transform.

Solution: If $Z[f(n)] = F(z)$ then $\lim_{z \to \infty} F(z) = f(0) = \lim_{t \to 0} f(t)$

We know that

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$$

$$= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

$$F(z) = f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots$$

$$\lim_{z \to \infty} F(z) = f(0) \left[Since \lim_{z \to \infty} \frac{1}{z} = 0\right]$$

$$\lim_{z \to \infty} F(z) = f(0) = \lim_{t \to 0} f(t)$$

8. Find the Z-transform of $(\boldsymbol{n}+\mathbf{1})(\boldsymbol{n}+\mathbf{2})$

Solution: $Z[(n+1)(n+2)] = Z[n^2 + 3n + 2]$

$$= Z(n^2) + 3Z(n) + Z(2)$$

$$= \frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{z-1}$$

9. Find the Z-transform of $\cos \boldsymbol{nx}$

Solution: $Z[e^{inx}] = Z[(e^{ix})^n]$

$$= \frac{z}{z - e^{ix}} \left[Since\ Z[a^n] = \frac{z}{z-a}\right]$$

$$= \frac{z}{z - (\cos\ x + i \sin x)} = \frac{z}{(z - \cos\ x) + i \sin x}$$

$$Z[\cos nx + isin\ nx] = \frac{z[(z - \cos\ x) - i \sin x]}{(z - \cos\ x)^2 + \sin^2 x}$$

$$Z[\cos nx] + iZ[sin\ nx] = \frac{z(z - \cos\ x)}{z^2 - 2zcos\ x + 1} - \frac{isin\ x}{z^2 - 2zcos\ x + 1}$$

Equating real and imaginary parts we get

$$Z(\cos nx) = \frac{z(z - \cos\ x)}{z^2 - 2zcos\ x + 1}$$

10. Find the Z-transform of $\boldsymbol{a^n}$

Solution:

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

$$= 1 + az^{-1} + (az^{-1})^2 + (az^{-1})^3 + \cdots [Since\ 1 + x + x^2 + x^3 + \cdots = (1-x)^{-1}]$$

$$= \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

11. Let $Z[f(n)] = F(z)$ and $k > 0$ then, Prove that

$$\boldsymbol{Z[f(n-k)] = z^{-k}F(z)}$$

Solution:

$$Z[f(n-k)] = \sum_{n=0}^{\infty} f(n-k)z^{-n} = \sum_{n=k}^{\infty} f(n-k)z^{-n}$$

$$[Since\ f(-1), f(-2), \ldots are\ not\ defined]$$

$$= f_0 z^{-k} + f_1 z^{-(k+1)} + f_2 z^{-(k+2)} + \cdots$$

$$= z^{-k}(f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots)$$

$$= z^{-k} \sum_{n=0}^{\infty} f(n) z^{-n}$$

$$= z^{-k}F(z)$$

12. Let $Z[f(n)] = F(z)$ and $k > 0$ then, Prove that

$$\boldsymbol{Z[f(n+k)] = z^{k}\left[F(z) - f(o) - f(1)z^{-1} - \cdots\ f(k-1)z^{-(k-1)}\right]}$$

Solution:

$$Z[f(n+k)] = \sum_{n=0}^{\infty} f(n+k)z^{-n}$$

Multiply and divide by z^k in RHS, we get

$$= \sum_{n=0}^{\infty} f(n+k)z^{-n}\frac{z^k}{z^k}$$

$$= z^k \sum_{n=0}^{\infty} f(n+k)z^{-(n+k)}$$

$$= z^k\left(f(k)z^{-k} + f(k+1)z^{-(k+1)} + f(k+2)z^{-(k+2)} + \cdots\right)$$

$$= z^k\left[\sum_{n=0}^{\infty} f(n)z^{-n} - \sum_{n=0}^{k-1} f(n-k)z^{-n}\right]$$

$$= z^k\left[F(z) - f(o) - f(1)z^{-1} - \cdots\ f(k-1)z^{-(k-1)}\right]$$

13. Find the Z-transform of $(-1)^n$

Solution:

$$Z[(-1)^n] = \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

$$= 1 - z^{-1} + z^{-2} - z^{-3} + \cdots$$

$$= (1 + z^{-1})^{-1}$$

$$= \frac{z}{z+1}$$

14. Find the Z-transform of n^2

Solution:

$$Z[n^2] = Z[n.n]$$

$$= -z\frac{dZ[n]}{dz} \qquad \left[Since\ Z[nf(t)] = -z\frac{dF(z)}{dz}\right]$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\frac{(z-1)^2 - 2(z-1)z}{(z-1)^4}$$

$$= -z\frac{(z-1) - 2z}{(z-1)^3}$$

$$= \frac{z(z+1)}{(z-1)^3}$$

15. State convolution theorem for Z-transform

Solution: If $f(n)\ and\ g(n)$ are two casual sequences,

$$Z[\{f(n) * g(n)\}] = Z\{f(n)\}.Z\{g(n)\} = F(z).G(z)$$

16. **If $F(z) = \frac{z}{z-e^{-T}}$, find $\lim_{t\to\infty} f(t)$**

Solution: We know that $\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)F(z)$

$$= \lim_{z\to 1}(z-1)\frac{z}{z-e^{-T}} = 0$$

17. Find the Z-transform of $nf(t)$ if $Z[f(t)] = F(z)$

Solution:

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n}$$

Differentiate both sides w.r.t. z^{-1}.

$$\frac{dF(z)}{dz} = \sum_{n=0}^{\infty} -nf(nT)z^{-n-1}$$

$$z\frac{dF(z)}{dz} = -\sum_{n=0}^{\infty} nf(nT)z^{-n} = -Z[nf(t)]$$

$$Z[nf(t)] = -z\frac{dF(z)}{dz}$$

18. If $Z[f(t)] = F(z)$ then prove that $Z[e^{-at}f(t)] = F(ze^{aT})$

Solution:

$$Z[e^{-at}f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(ze^{aT})^{-n}$$

$$= F(ze^{aT}) \qquad \left[Since\ F(z) = \sum_{n=0}^{\infty} f(nT)z^{-n}\right]$$

19. Find the Z-transform of $e^{-at}\cos bt$

Solution:

$$Z[e^{-at}\cos bt] = \{Z[\cos bt]\}_{z\to ze^{aT}} \qquad [Since \quad Z[e^{-at}f(t)] = F(ze^{aT})]$$

$$= \left\{\frac{z(z-\cos bT)}{z^2 - 2z\cos bT + 1}\right\}_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}(ze^{aT} - \cos bT)}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$$

20. If $F(z) = \frac{10z}{(z-1)(z-2)}$ find $f(0)$.

Solution:

$$f(0) = \lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{10z}{(z-1)(z-2)}$$

$$= \lim_{z\to\infty} \frac{10z}{z\left(1-\frac{1}{z}\right)(z-2)}$$

$$= \lim_{z\to\infty} \frac{10}{\left(1-\frac{1}{z}\right)(z-2)} = 0$$

21. Find Z-transform of $\frac{1}{n}, n \geq 1$.

Solution:

$$Z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$$

$$= -\log\left(1 - \frac{1}{z}\right) \; if \; \left|\frac{1}{z}\right| < 1$$

$$= \log\left(\frac{z}{z-1}\right) \; if \; |z| > 1$$

22. Find Z-transform of $\frac{1}{n!}$

Solution:

$$Z\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = \sum_{n=0}^{\infty} \frac{1}{n!} (z^{-1})^n$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{(z^{-1})^2}{2!} + \frac{(z^{-1})^3}{3!} + \cdots = e^{z^{-1}}$$

$$Z\left[\frac{1}{n!}\right] = e^{1/z}$$

23. Find the Z-transform of ab^n

Solution:

$$Z[\, ab^n] = \sum_{n=0}^{\infty} ab^n z^{-n}$$

$$= a \sum_{n=0}^{\infty} \left(\frac{b}{z}\right)^n$$

$$= a\left[1 + \frac{b}{z} + \left(\frac{b}{z}\right)^2 + \cdots\right]$$

$$= a \frac{1}{1 - \frac{b}{z}} \; if \; \left|\frac{b}{z}\right| < 1$$

$$= \frac{az}{z-b} \; if \; |z| > |b|$$

24. Find the inverse Z-transform by using convolution theorem

$$\frac{z^2}{(z-a)^2}$$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a} . \frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right] = a^n * a^n$$

$$= \sum_{k=0}^{n} a^{n-k}.a^k = \sum_{k=0}^{n} a^n$$

$$= (n+1)a^n u(n)$$

25. Define Z-transform

Solution:

The two sided Z-transform $F(z)$ for a sequence $f(x)$ is defined as

$$F(z) = \sum_{n=-\infty}^{\infty} f(x)z^{-n}$$

The one sided Z-transform $F(z)$ for a sequence $f(x)$ is defined as

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26. Find the value of $Z[a^n f(x)]$

Solution:

$$Z[a^n f(x)] = \sum_{n=0}^{\infty} a^n f(x)z^{-n}$$

$$= \sum_{n=0}^{\infty} f(x)\left(\frac{z}{a}\right)^{-n}$$

$$= F\left[\frac{z}{a}\right] \qquad \left[Since\ F(z) = \sum_{n=0}^{\infty} f(x)z^{-n}\right]$$

27. Find the value of $Z[1]$.

Solution:

$$Z[1] = \sum_{n=0}^{\infty} 1.z^{-n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1} \qquad if\ \left|\frac{1}{z}\right| < 1$$

$$= \frac{z}{z-1}\ if\ |z| > 1.$$

28. Find the Z-transform of $\delta(n-k)$

Solution:

$$Z[\delta(n-k)] = \sum_{n=0}^{\infty} \delta(n-k) z^{-n}$$

$$= \frac{1}{z^k}, \text{if k is positive integer}$$

29. Find the Z-transform of $\boldsymbol{u(n-1)}$

Solution:

$$Z[u(n-1)] = \sum_{n=1}^{\infty} 1.z^{-n}$$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} \qquad if \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z-1} \; if \; |z| > 1$$

30. Find the Z-transform of $\boldsymbol{3^n \delta(n-1)}$

Solution:

$$Z[3^n \delta(n-1)] = \{Z[\delta(n-k)]\}_{z \to \frac{z}{3}}$$

$$= \left\{\frac{1}{z}\right\}_{z \to \frac{z}{3}} = \frac{3}{z}$$

Applications of partial differential equations

Part-A

1. Classify the partial differential equation $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$.

Solution: Given $3u_{xx} + 4u_{xy} + 3u_y - 2u_x = 0$.

$$A = 3, B = 4, C = 0$$

$B^2 - 4AC = 16 > 0$, Hyperbolic

2. The ends A and B of a rod of length $10\ cm$ long have their temperature kept at $20^o\ C$ and $70^o\ C$. Find the steady state temperature distribution on the rod.

Solution: When the steady state condition exists the heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} = 0$$

i.e., $u(x) = c_1 x + c_2 \ldots (1)$

The boundary conditions are $(a)\ u(0) = 20, (b)\ u(10) = 70$

Applying (a) in (1), we get $u(0) = c_2 = 20$

Substituting $c_2 = 20$ in (1), we get

$$u(x) = c_1 x + 20 \ldots (2)$$

Applying (b) in (2), we get $u(10) = c_1 10 + 20 = 70 \Rightarrow c_1 = 5$

Substituting $c_1 = 5$ in (2), we get

$$u(x) = 5x + 20$$

3.Write the one dimensional wave equation with initial and boundary conditions in which the initial position of the string is $f(x)$ and the initial velocity imparted at each point x is $g(x)$.

Solution: the one dimensional wave equation is

$$\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

$$(i)\ y(0,t) = 0 \qquad (ii)\ y(l,t) = 0, \qquad (iii)\ y(x,0) = f(x) \quad (iv)\ \frac{\partial y\ (x,0)}{\partial t} = g(x)$$

4.In steady state conditions derive the solution of one dimensional heat flow equation.

Solution: when steady state conditions exist the heat flow equation is independent of time t.

$$\frac{\partial u}{\partial t} = 0$$

The heat flow equation becomes

$$\frac{\partial^2 u}{\partial x^2} = 0$$

The solution of heat flow equation is $u(x) = c_1 x + c_2$

5. What is the basic difference between the solutions of one dimensional wave equation and one dimensional heat equation.

Solution: Solution of the one dimensional wave equation is of periodic in nature. But solution of one dimensional heat equation is not of periodic in nature.

6. What are the possible solutions of one dimensional wave equation?

Solution:

$$y(x,t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{pat} + c_4 e^{-pat}) \ldots (1)$$

$$y(x,t) = (c_5 \cos px + c_6 \sin px)(c_7 \cos pat + c_8 \sin pat) \ldots (2)$$

$$y(x,t) = (c_9 x + c_{10})(c_{11} t + c_{12}) \ldots (3)$$

7. Write down possible solutions of the Laplace equation.

Solution:

$$u(x,y) = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \ldots (1)$$

$$u(x,y) = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8 e^{-py}) \ldots (2)$$

$$u(x,y) = (c_9 x + c_{10})(c_{11} y + c_{12}) \ldots (3)$$

8. State Fourier law of conduction.

Solution:

Fourier law of heat conduction:

The rate at which heat flows across an area A at a distance x from one end of a bar is given by

$$Q = -KA\left(\frac{\partial u}{\partial x}\right)_x$$

where K is the thermal conductivity and $\left(\frac{\partial u}{\partial x}\right)_x$ means the temperature gradient at x.

9. In the wave equation $\frac{\partial^2 y}{\partial t^2} = \alpha^2 \frac{\partial^2 y}{\partial x^2}$ what does α^2 stands for?

Solution:

$$\alpha^2 = T = \frac{Tension}{mass}$$

10. State any two laws which are assumed to derive one dimensional heat equation.

Solution:

(i) The sides of the bar are insulated so that the loss or gain of heat from the sides by conduction or radiation is negligible.

(ii) The same amount of heat is applied at all points of the face.

11. Classify the partial differential equation $u_{xx} + xu_{xy} = 0$.

Solution:

Here $A = 1, B = x, C = 0$

$$B^2 - 4AC = x^2$$

$$(i)\ Parabolic\ \ if\ x = 0.$$

$$(ii)\ Hyperbolic\ \ if\ x > 0\ and\ x < 0.$$

12. A rod 30 cm long has its ends A and B kept at 20^o C and 80^o C respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution: Let $l = 30\ cm$

When the steady state condition prevail the heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} = 0$$

i.e., $u(x) = ax + b \ldots (1)$

When the steady state condition exists the boundary conditions are

$$u(0) = 20\ , u(l) = 80 \ldots (2)$$

Applying (2) in (1) we get

$$u(0) = b = 20 \ldots (3)$$

and $u(l) = al + 20 = 80$

$$a = \frac{60}{l} = \frac{60}{30} = 2 \ldots (4)$$

Substituting (3) and (4) in (1) we get

$$u(x) = 2x + 20$$

13. Classify the partial differential equation

$$\boldsymbol{(a)\ y^2\, u_{xx} - 2xyu_{xy} + x^2\, u_{yy} + 2u_x - 3u = 0}$$

$$\boldsymbol{(b)\ y^2\, u_{xx} + u_{yy} + u_x^2 + u_y^2 + 7 = 0}$$

Solution:

a) $A = y^2, B = -2xy, C = x^2$

$$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$$

Parabolic

b) $A = y^2, B = 0, C = 1$

$$B^2 - 4AC = -4y^2 < 0$$

Elliptic.

14. A rod 60 cm long has its ends A and B kept at 30^o C and 40^o C respectively until steady state conditions prevail. Find the steady state temperature in the rod.

Solution: Let $l = 60\ cm$

When the steady state condition prevail the heat flow equation is

$$\frac{\partial^2 u}{\partial x^2} = 0$$

i.e., $u(x) = ax + b \ldots (1)$

When the steady state condition exists the boundary conditions are

$$u(0) = 30\ , u(l) = 40 \ldots (2)$$

Applying (2) in (1) we get

$$u(0) = b = 30 \ldots (3)$$

and $u(l) = al + 30 = 40$

$$a = \frac{10}{l} = \frac{10}{60} = \frac{1}{6} \dots (4)$$

Substituting (3) and (4) in (1) we get

$$u(x) = \frac{1}{6}x + 20$$

15. Classify the partial differential equation

$$\boldsymbol{a)\ 4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} - 8\frac{\partial u}{\partial y} - 16u = 0}$$

$$\boldsymbol{b)\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

Solution:

a) $A = 4, B = 4, C = 1$

$$B^2 - 4AC = 16 - 16 = 0$$

Parabolic.

b) $A = 1, B = 0, C = 1$

$$B^2 - 4AC = -4 < 0$$

Elliptic.

16. Classify the partial differential equation

$$\boldsymbol{a)\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \qquad b)\ \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + xy}$$

Solution:

a) Here $A = 1, B = 0, C = -1$

$$B^2 - 4AC = 4 > 0$$

Hyperbolic.

b) Here $A = 0, B = 1, C = 0$

$$B^2 - 4AC = 1 > 0$$

Hyperbolic.

17. State one dimensional heat equation with the initial and boundary conditions.

Solution: The one dimensional heat equation is

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

Initial condition means condition at $t = 0$

Boundary condition means condition at $x = 0 \; and \; x = l$

18. Write the boundary conditions and initial conditions for solving the vibration of string equation, if the string is subjected to initial displacement $f(x)$ and initial velocity $g(x)$.

Solution:

Initial displacement $f(x)$

$$(i) y = 0 \quad when \; x = 0$$

$$(ii) y = 0 \quad when \; x = l$$

$$(iii) \frac{\partial y}{\partial t} = 0 \quad when \; t = 0$$

$$(iv) y = f(x) \quad when \; t = 0$$

Initial velocity $g(x)$

$$(i) y = 0 \quad when \; x = 0$$

$$(ii) y = 0 \quad when \; x = l$$

$$(iii) \frac{\partial y}{\partial t} = g(x) \quad when \; t = 0$$

$$(iv) y = 0 \quad when \; t = 0$$

19. Write down the differential equation for two dimensional heat flow equation for the unsteady state.

Solution: The two dimensional heat flow equation for the unsteady state is given by

$$\alpha^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial u}{\partial t}$$

20. What is the steady state heat flow equation in two dimensions in Cartesian form?

Solution: The two dimensional heat flow equation for the steady state is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. [Laplace\ equation]$$

21. Write down the polar form of two dimensional heat flow equation in steady state.

Solution:

$$r^2 \frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0. [Laplace\ equation]$$

22. Write all the solutions of Laplace equation in polar coordinates.

Solution:

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \dots (1)$$

$$u(r,\theta) = \big(c_5 \cos(plog\, r) + c_6 (plog\, r)\big)\big(c_7 e^{p\theta} + c_8 e^{-p\theta}\big) \dots (2)$$

$$u(r,\theta) = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \dots (3)$$

23. Explain the initial and boundary value problems.

Solution: The values of a required solution, on the boundary of some domain will be given. These are called boundary conditions. In other cases, when time t is one of the variables, the values of the solution at t=0 may be presented. These are called initial conditions.

The partial differential equation together with these conditions constitutes a boundary value problem or an initial value problem, according to the nature of the condition.

24. What is meant by steady state condition in heat flow?

Solution:

Steady state condition in heat flow means that the temperature at any point in the body does not vary with time. i.e., it is independent of time t.

25. Distinguish between steady and unsteady states in heat conduction problems.

Solution:

In unsteady state, the temperature at any point of the body depends on the position of the point and also the time t. In steady state, the temperature at any point depends only on the position of the point and is independent of the time t.

26. Solve using separation of variables method $\boldsymbol{yu_x + xu_y = 0}$.

Solution:

Given $yu_x + xu_y = 0 \dots (1)$

Let $u = X(x).Y(y) \dots (2)$

be the solution of (1). From (2) we get

$$u_x = X^{'}Y \dots (3)$$

$$u_y = XY' \dots (4)$$

Substituting (3) and (4) in (1), we get

$$y.X^{'}Y + x.XY^{'} = 0$$

$$y.X^{'}Y = -x.XY^{'}$$

$$\frac{X'}{xX} = -\frac{Y'}{yY} = k$$

$$X^{'} = kxX \qquad or \qquad Y^{'} = -kyY$$

$$\frac{dX}{dx} = kxX \qquad or \qquad \frac{dY}{dy} = -kYy$$

$$\frac{dX}{X} = kx\,dx \qquad or \qquad \frac{dY}{Y} = -ky\,dy$$

$$\log X = k\frac{x^2}{2} + k_1 \qquad or \qquad \log Y = -k\frac{y^2}{2} + k_2$$

$$X = c_1 e^{k\frac{x^2}{2}} \qquad or \qquad Y = c_2 e^{-k\frac{y^2}{2}} \dots (5)$$

Substituting (5) in (2) we get

$$u(x,y) = c_1 c_2 e^{\frac{k(x^2 - y^2)}{2}}$$

27. By the method of separation of variables solve $x^2 q + y^3 p = 0$

Solution: Given

$$x^2 \frac{\partial z}{\partial y} + y^3 \frac{\partial z}{\partial x} = 0 \dots (1)$$

Let $z = X(x)Y(y) \dots (2)$ be the solution of (1).

Then

$$z_x = X^{'}Y \dots (3)$$

$$z_y = XY' \dots (4)$$

Substituting (3) and (4) in (1), we get

$$y^3.X'Y + x^2.XY' = 0$$

$$-y^3.X'Y = x^2.XY'$$

$$\frac{X'}{x^2X} = -\frac{Y'}{y^3Y} = k$$

$$X' = kx^2X \qquad or \qquad Y' = -ky^3Y$$

$$\frac{dX}{dx} = kx^2X \qquad or \qquad \frac{dY}{dy} = -kYy^3$$

$$\frac{dX}{X} = kx^2\,dx \qquad or \qquad \frac{dY}{Y} = -ky^3\,dy$$

$$\log X = k\frac{x^3}{3} + k_1 \qquad or \qquad \log Y = -k\frac{y^4}{4} + k_2$$

$$X = c_1e^{k\frac{x^3}{3}} \qquad or \qquad Y = c_2e^{-k\frac{y^4}{4}} \ldots (5)$$

Substituting (5) in (2) we get

$$u(x,y) = c_1c_2e^{k\frac{x^3}{3}}e^{-k\frac{y^4}{4}}$$

28. State the assumptions made in the derivation of one dimensional wave equation.

Solution:

(i) The mass of the string per unit length is constant

(ii) The string is perfectly elastic and does not offer any resistance to bending

(iii) The tension caused by stretching the string before fixing it at the end points is so large that the action of the gravitational force on the string can be neglected.

(iv) The string performs a small transverse motion in a vertical plane, that is every particle of the string moves strictly vertically so that the deflection and the slope at every point of the string remain small in absolute value.

29. Write down the boundary condition for the following boundary value problem " If a string of length 'l' initially at rest in its equilibrium position and each of its point is given the velocity $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 sin^3\frac{\pi x}{l}$ $0 < x < l$, determine the displacement function $y(x,t)$"?

Solution: The boundary conditions are

$$i)\ y(0,t) = 0, t > 0 \qquad ii)\ y(l,t) = 0, t > 0$$

$iii)\ y(x,0) = 0, 0 < x < l$ $\qquad$ $iv)\ \frac{\partial y(x,0)}{\partial t} = v_0 sin^3 \frac{\pi x}{l},\ \ 0 < x < l$

30. If the ends of a string of length 'l' are fixed and the midpoint of the string is drawn aside through a height 'h' and the string is released from rest, write the initial conditions.

Solutions:

$$(i) y(0,t) = 0$$

$$(ii) y(l,t) = 0$$

$$(iii) \frac{\partial y(x,0)}{\partial t} = 0$$

$$(iv)\ y(x,0) = \begin{cases} \frac{2hx}{l}, & 0 < x < \frac{l}{2} \\ \frac{2h}{l}(l-x), & \frac{l}{2} < x < l \end{cases}$$