SRI RAMAKRISHNA INSTITUTE OF TECHNOLOGY, COIMBATORE- 10

DEPARTMENT OF SCIENCE AND HUMANITIES

SUBJECT: NUMERICAL METHODS & LINEAR PROGRAMMING

UNIT II

SOLUTIONS OF EQUATION

1. If $g(x)$ is continuous in $[a, b]$, then under what condition the iterative method $x = g(x)$ has a unique solution in$[a, b]$?

Solution : $|g'(x)| < 1 \quad in \quad [a, b]$

2. State the iterative formula for Regula Falsi method to solve $f(x) = 0$.

Solution:

The iteration formula to find the root of the equation $f(x) = 0.$ which lies between $x = a \quad and \quad x = b$ is

$$x_1 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)}$$

3. State the iterative formula for solving $f(x) = 0$ by Bisection method.

Solution :

The iteration formula for to find the root of the equation $f(x) = 0$ which lies between $x = a \quad and \quad x = b$ is $x_i = \frac{a+b}{2}$.

4. Explain Regula Falsi method of getting a root.

Solution :

1. Find two numbers ' a ' and ' b ' such that $f(a) \ and \ f(b)$ are of opposite signs. Then a root lies between ' a ' and ' b '.
2. The first approximation to the root is given by $x_1 = \frac{a\ f(b) - b\ f(a)}{f(b) - f(a)}$.
3. If $f(x_1) \ and \ f(a)$ are opposite signs, then the actual root lies between 'x_1 ' and ' a '. Now replacing ' b ' by 'x_1 ' and keeping ' a ' as it is we get the closer approximation ' x_2' to the actual root.
4. This procedure is repeated till the root is found to the desired degree of accuracy.

5. How to reduce the number of iterations while finding the root of an equation by Regula Falsi method.

Solution :

The number of iterations to get a good approximation to the real root can be reduced, if we start with a smaller interval for the root.

6. What is the order of convergence of Newton- Raphson method if the multiplicity of the root is one.

Solution : The order of convergence of Newton-Raphson method is 2

7. What is the rate of convergence in Newton-Raphson method ?

Solution : The rate of convergence in Newton-Raphson method is 2.

8. What is the criterion for convergence of Newton- Raphson –method ?

Solution :

The Criterion for convergence of Newton- Raphson -method is

$$|f(x)\, f''(x)| < |f'(x)|^2 \quad \textit{in the interval considered.}$$

9. Write the iterative formula for Newton- Raphson –method.

Solution:

The Newton- Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

10. What is the condition of convergence of a fixed point iteration method?

Solution :

Let $f(x) = 0$ be the given equation whose actual root is r. The equation $f(x) = 0$ be written as $x = g(x)$. Let I be the interval containing the root $x = r$. If $|g'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, \dots \dots . x_n$ will converge to r, if the initial starting value x_0 is chosen in I.

11. If $g(x)$ is continuous in $[a, b]$, then under what condition the iterative method $x = g(x)$ has a unique solution in$[a, b]$?

Solution: $|g'(x)| < 1 \quad in \quad [a, b]$

12. What is the order of convergence of fixed point iteration method.

Solution: The order of convergence of Newton-Raphson method is 1.

13. In what form is the coefficient matrix transformed into when $A\,X = B$ is solved by Gauss – elimination method?

Solution: Upper triangular matrix.

14. In what form is the coefficient matrix transformed into when $A\,X = B$ is solved by Gauss – Jordan method?

Solution: Diagonal matrix.

15. Explain briefly Gauss – Jordan iteration to solve simultaneous equation?

Solution :

Consider the system of equations $A\,X = B$.
If A is a square matrix the given system reduces to

$$\begin{pmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{21} & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & a_{nn} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix}$$

This system is reduces to the following n equations.

$$a_{11}x_1 = b_1, \quad a_{22}x_2 = b_2, \ldots\ldots\ a_{nn}x_n = b_n$$

$$x_1 = \frac{b_1}{a_{11}}, \quad x_2 = \frac{b_2}{a_{22}}, \ldots\ldots\ x_n = \frac{b_n}{a_{nn}}$$

The method of obtaining the solution of the system of equations by reducing the matrix A to diagonal matrix is known as Gauss – Jordan elimination method.

16. For solving a linear system, compare Gauss - elimination method and Gauss – Jordan method.
Solution :

	Gauss - elimination method	Gauss – Jordan method
1	Coefficient matrix transformed into upper triangular matrix	Coefficient matrix transformed into diagonal matrix
2	Direct method	Direct method
3	We obtain the solution by Backward substitution method	No need of Backward substitution method

17. State the principle used in Gauss – Jordan method.
Solution:
Coefficient matrix transformed into upper triangular matrix

18. Write the sufficient condition for Relaxation method to converge.
Solution:
The coefficient of matrix should be diagonally dominant.

$$|a_1| \geq |b_1| + |c_1|, \; |b_2| \geq |a_2| + |c_2|, \quad |c_3| \geq |a_3| + |b_3|$$

19. Write the sufficient condition for Gauss - Siedal method to converge.
Solution:
The coefficient of matrix should be diagonally dominant.

$$|a_1| > |b_1| + |c_1|, \; |b_2| > |a_2| + |c_2|, \quad |c_3| > |a_3| + |b_3|$$

20. State the sufficient condition for Gauss - Jacobi method to converge.

Solution:
The coefficient of matrix should be diagonally dominant.

$$|a_1| > |b_1| + |c_1|,\ |b_2| > |a_2| + |c_2|, \quad |c_3| > |a_3| + |b_3|$$

21. Give two indirect methods to solve a system of linear equations.
Solution:
(1). Gauss – Jacobi method
(2). Gauss – Siedal method

22. Compare Gauss – Jacobi method and Gauss – Siedal method
Solution:

	Gauss – Jacobi method	Gauss – Siedal method
1	Convergence rate is slow	The rate of convergence of Gauss – Siedal method is roughly twice that of Gauss - Jacobi
2	Indirect method	Indirect method
3	Condition for the convergence is the coefficient matrix is diagonally dominant	Condition for the convergence is the coefficient matrix is diagonally dominant

23. Find the first approximation to the root lying between 0 and 1 of $x^3 + 3x - 1 = 0$ by Newton's method.
Solution: $f(x) = x^3 + 3x - 1 \ \ and \ \ f'(x) = 3x^2 + 3$

$$x_1 = x_n - \frac{f(x_0)}{f'(x_0)} \implies 0 - \left(\frac{-1}{3}\right) = 0.3333.$$

24. Find an iteration formula for finding the square root of N by Newton method.
Solution: $f(x) = x^2 - N \ \ and \ \ f'(x) = 2\,x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\implies x_n - \frac{({x_n}^2 - N)}{2\ x_n} = \left(\frac{1}{2}\right)\left(x_n + \frac{N}{x_n}\right), n = 0,1,2, \ldots$$

25. How do you express the equation $x^3 + x^2 - 1 = 0$ for the positive root by iteration method?
Solution:

$$x^3 + x^2 - 1 = 0 \implies x^2(x+1) - 1 = 0$$

$$\implies x^2(x+1) = 1 \implies x^2 = \frac{1}{(x+1)}$$

$$x = \frac{1}{\sqrt{x+1}} = g(x)$$

26. Can we write iteration method to find the root of the equation $2x - \cos x - 3 = 0 \ \ in$ $\left[0, \frac{\pi}{2}\right]$?.
Solution:

$$2x - \cos x - 3 = 0 \implies 2\,x = 3 + \cos x$$

$$\Rightarrow \quad x = \frac{1}{2}\,(3 + \cos x) = \phi(x)$$

$$\Rightarrow \quad \phi(x) = \frac{1}{2}\,(3 + \cos x)$$

$$\Rightarrow |\phi'(x)| = \left|\frac{\sin x}{2}\right|$$

$$\Rightarrow |\phi'(0)| = \left|\frac{\sin 0}{2}\right| 0 < 1 \quad and \quad \Rightarrow \left|\phi'\left(\frac{\pi}{2}\right)\right| = \left|\frac{\sin\frac{\pi}{2}}{2}\right| = 0.5 \ < 1.$$

Hence $|\phi'(x)| = \left|\frac{\sin x}{2}\right| < 1 \ for\ all\ \ I = \left[0, \frac{\pi}{2}\right]$. Therefore we can use iteration method.

27. Is the system of equations $3x + 9y - 2z = 10,\ 4x + 2y + 13\,z = 1,$
$4x - 2y + z = 3$ Diagonally dominant? If not make it diagonally dominant.
Solution :
Let $x + 9y - 2z = 10,\ 4x + 2y + 13\,z = 1, 4x - 2y + z = 3$

$|a_1| > |b_1| + |c_1|,\ \ |b_2| > |a_2| + |c_2| \ \ and \ \ |c_3| > |a_3| + |b_3|$

Hence the given system is not diagonally dominant.

Hence we rearrange the system as follows $\quad 4x - 2y + z = 3$

$$x + 9y - 2z = 10,$$

$$4x + 2y + 13\,z = 1$$

28. Explain power method to find the dominant Eigen value of a square matrix.
Solution :
If v_0 is an initial arbitrary vector, then compute $y_{k+1} = A\,v_k \ \ and \ \ v_{k+1} = \frac{Y_{k+1}}{m_{k+1}}$ where m_{k+1} is the numerically largest element of y_{k+1}. Then dominant Eigen value
$\lambda_1 = \lim_{n\to\infty} \frac{[Y_{k+1}]_i}{[v_k]_i}\ , i = 1,2, \dots n$
29. Write Newton's formula for finding cube root of N.
Solution :

$$x^3 - N = 0 \quad \Rightarrow \ f(x) = x^3 - N \ \ and \ \ f'(x) = 3\,x^2$$

$$By\ newton's method, we\ have \quad x_{n+1} = x_n - \frac{{x_n}^3 - N}{3\,{x_n}^2}\ ,\ \ n = 1,2, \dots$$

30. Write Newton's formula for finding reciprocal of a positive number N.
Solution :

$$N = \frac{1}{x} \quad \Rightarrow \ f(x) = \frac{1}{x} - N \ \ and \ \ f'(x) = \frac{-1}{x^2}$$

$$By\ newton'smethod \quad x_{n+1} = x_n - \frac{\left(\frac{1}{x_n} - N\right)}{\frac{-1}{{x_n}^2}} = x_n(2 - N\,x_n), \qquad n = 1,2,\ldots$$

Part – B

1. Find the positive root of $x^3 + x^2 - 100 = 0$ by Iteration method.
2. Find the positive root of $3x + \sin x = e^x$ by Bisection method.
3. Find the positive root of the equation $x^3 - x - 9 = 0$ by Iteration method lying between 2 & 3.
4. Find by Iteration method, the root of the equation $x = 0.21\sin(0.5 + x)$ near $x = 0$.
5. Using Newton's method to find the root of $x^3 = 6x - 4$ correct to three decimal places.
6. Using Newton's iterative method to find the root of $2x - \log_{10} x = 7$ correct to three decimal Places
7. Find by Newton's method, the root of the equation $x\,e^x = \cos x$.
8. Find by Newton's method, the root of the equation $x\sin x + \cos x = 0$.
9. Solve the system of equations by Relaxation method.
$2x + 4y + 8z = 41,\ \ 4x + 6y + 10\,z = 56,\quad 6x + 8y + 10z = 64$.
10. Solve the system of equations by Gauss elimination method.
$5x_1 + x_2 + x_3 + x_4 = 4,\quad x_1 + 7x_2 + x_3 + x_4 = 12,\quad x_1 + x_2 + 6\,x_3 + x_4 = -5,$
$5x_1 + x_2 + x_3 + 4x_4 = -6,$.
11. Solve the system of equations by Crout's Jordan method.
$3x + 4y + 5z = 18,\ \ 2x - y + 8\,z = 13,\ \ 5x - 2y + 7z = 20.$
12. Solve the system of equations by Gauss - Jordan method.
$x + 2y + z = 8,\ \ 2x + 3y + 4z = 20\ \ and\quad 4x + y + 2z = 12.$
13. Solve the system of equations by Gauss - Jacobi method.
$10x - 2y + z = 12,\ \ x + 9y - z = 10\ \ and\ \ 2x - y + 11\,z = 20$
14. Solve the system of equations by Gauss - Jacobi method.
$83x + 11y - 4z = 95 = 95,\ \ 7x + 52y + 13z = 104\ \ and\ \ 3x + 8y + 29z = 71$
15. Solve the system of equations by Gauss – Siedal method.
$4x - y - z = -7,\ \ x - 5y + z = -10\ \ and\ \ x + 2y + 6z = 9$
16. Solve the system of equations by Relaxation method.
$2x + 5y - z = 10,\ \ x + 3y + 6z = -1\ \ and\ \ 8x - y + 3z = 12.$