SRI RAMAKRISHNA INSTITUTE OF TECHNOLOGY, COIMBATORE- 10

DEPARTMENT OF SCIENCE AND HUMANITIES

SUBJECT: NUMERICAL METHODS & LINEAR PROGRAMMING

UNIT - V

01. Write down the order Taylor's algorithim.

Answer :

Let $y' = f(x, y)\ \ with\ \ y(x_0) = y_0$

Then the Taylor algorithim is given by

$$y(x_1) = y_0 + \frac{(x - x_0)}{1!}\, y_0' + \frac{(x - x_0)^2}{2!}\, y_0'' + \frac{(x - x_0)^3}{3!}\, y_0''' + \frac{(x - x_0)^4}{4!}\, y_0^{\prime iv} + \cdots$$

$$where\ \ x_1 = x_0 + h\ \ \ and\ \ y_0^{(r)} = \frac{d^r y}{dx^r}\ \ at\ \ (x_0, y_0).$$

02. What are the merits and demerits of the Taylor series method of solution?

Answer :

It is a powerful single step method.

It is the best method if the expression for higher order derivtives are simpler.

The major demerit of this method is the evaluation of higher order derivatives become tedious for complicated algebric expressions.

03. Given $y' = x + y,\ y(0) = 1.$ Find $y(0.1)$ By Taylor's method.

Answer :

$$y' = x + y\,;\ \ y'' = 1 + y';\ \ y''' = y'' \ldots\ldots$$

$$x_0 = 0, \qquad y_0 = 0.\ Then\ \ y(0.1) = y_1 = y_0 + \frac{h}{1!}\, y_0' + \frac{h^2}{2!}\, y_0'' + \cdots$$

$$when\ \ h = 0.1. \qquad \therefore\ \ y(0.1) = 1 + 0.1 + \frac{0.01}{2}\,(2) + \frac{0.001}{6}\,6\,(2)$$

$$\Longrightarrow\ \ y(0.1) = 1.1103$$

04. Find $y(0.1)\ by$ Euler's method, given that $y' = 1 - y,\ y(0) = 0.$

Answer :

$$y_1 = y_0 + h\, f(x_0, y_0) = 0.01[1 - 0] = 0.1 \quad \Longrightarrow\ \ y(0.1) = 0.1$$

05. Using Euler's method compute for $x = 0.1\ \ \&\ \ 0.2\ \ with\ \ h = 0.1\ \ \ given$ $y' = y - \frac{2x}{y}\,, y(0) = 1.$

Answer :

$$y_1 = 1 + 0.1[1 - 0] = 1.10 \quad \Longrightarrow\ \ y(0.1) = 1.10$$

$$y_2 = 1.1 + 0.1\left[1.1 - \frac{0.2}{1.1}\right] = 1.19 \quad \Rightarrow \quad y(0.2) = 1.19$$

06. Find $y(0.1)$ *by* Euler's method, given that $y' = x + y,\ y(0) = 1.$

Answer :

$$y_1 = 1 + 0.1[0 + 1] = 1.10 \quad \Rightarrow \quad y(0.1) = 1.10$$

07. Given $y' + y = 0$ *and* $y(0) = 1.$ Find $y(0.01)$ *and* $y(0.02)$ by Euler's method

Answer :

$$y_1 = y_0 + h\, f(x_0, y_0) = 0.01[1 - 0] = 0.1 \quad \Rightarrow \quad y(0.1) = 0.1$$

08. Using Euler's method compute for $x = 0.1\ \&\ 0.2$ *with* $h = 0.1$ *given*

$y' = y - \frac{2x}{y}, y(0) = 1.$

Answer :

$$y_1 = 1 + 0.01[-1] = 0.09 \quad \Rightarrow \quad y(0.01) = 0.09$$

$$y_1 = 0.99 + 0.01[-0.99] = 0.9801 \quad \Rightarrow \quad y(0.02) = 0.9801$$

09. State the algorithim for modified Euler's method, to solve $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0.$

Answer :

$$y_{n+1}^{(1)} = y_n + h\, f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f\left(x_{n+1}, y_{n+1}^{(1)}\right)\right]$$

$$where \quad n = 0,1,2, \ldots \quad and \quad x_{n+1} = x_n + h$$

10. State Rung-Kutta fourth order formulae for solving $\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0.$

Answer :

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \quad where$$

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

11. What are th distinguish property for Rung-Kutta methods?

Answer :

(1). These methods do not require the higher order derivatives and requires only the function values at different points.

(2). To evaluate y_{n+1}, we need only y_n but not previous of y's.

(3). The solution by these methods agree with Taylor series solution upto the terms of h^r Where r is the order of Runge-Kutta method.

12. Which is the better Taylor series method or Runge – Kutta method? Why?

Answer :

Runge-Kutta method is better since higher order derivatives of y are not required. Taylor's series method involves use of higher oder derivatives which may be difficult in case of complicated algebric functions.

13. State the order of error in R-K method of fourth order.

Answer :

Error of fourth order method is $O(h^2)$ where h is the interval of differencing.

14. State Milne's Predictor and Corrector formula.

Answer :

Predictor : $y_{n+1}^{p} = y_{n-3} + \frac{4h}{3}\,[2\,y'_{n-2} - y'_{n-1} + 2\,y'_n]$

Corrector : $y_{n+1}^{c} = y_{n-1} + \frac{h}{3}\,[\,y'_{n-2} + 4y'_{n-1} + \ y'_{n+1}]$

15. State Adam's Predictor and Corrector formula.

Answer :

Predictor : $y_{n+1,P} = y_n + \frac{h}{24}\,[55\,y'_n - 59y'_{n-1} + 37\,y'_{n-2} - 9y'_{n-3}]$

Corrector : $y_{n+1,C} = y_n + \frac{h}{24}\,[9\,y'_{n+1} + 19y'_n - 5\,y'_{n-1} + y'_{n-2}]$

16. Write the predictor error and corrector error in Milne's method.

Answer :

Predictor error $= \frac{14}{45}\,h^5 f^{iv}(\mathcal{E})$ Corrector error $= -\frac{h^5}{90}\,y^{iv}(\mathcal{E})$

17. Distinguish Single – step and Multi – step methods.

Answer :

Single – step methods : To find y_{n+1} , the information at y_n is enough.

Multi – step methods : To find y_{n+1} , the past four values $y_{n-3}, y_{n-2}, y_{n-1}$ and y_n are needed.

18. State the conditions for the equation.

$A\,u_{xx} + B\,u_{xy} + C\,u_{yy} + D\,u_x + E\,u_y + F\,u = G$ where A, B, C, D, E, F, G are function of x and y to be (i) . Elliptic (ii). Parabolic (iii). Hyperbolic.

Solution :

The given equation is said to be

(i). Elliptic at appoint (x, y) in the plane if $B^2 - 4\,A\,C < 0$

(ii). Parabolic if $B^2 - 4\,A\,C = 0$

(iii). Hyperbolic if $B^2 - 4\,A\,C > 0$.

19. State the conditions for the equation.

$A\,u_{xx} + 2\,B\,u_{xy} + C\,u_{yy} = f(u_x, u_y, x, y)$ to be

(i) . Elliptic (ii). Parabolic (iii). Hyperbolic when A, B, C are functions of x and y.

Solution :

The given equation is elliptic if $(2\,\mathrm{B}^2) - 4\,A\,C < 0$

$$i,e\,, \qquad B^2 - A\,C < 0.$$

The given equation is parabolic if $\mathrm{B}^2 - A\,C = 0$.

The given equation is hyperbolic if $\mathrm{B}^2 - A\,C > 0$.

20. Fill up the blank.

The equation $y\,u_{xx} + u_{yy} = 0$ is hyperbolic in the region ………………………

Solution :

Here $A = y, B = 0, C = 1$

$$\mathrm{B}^2 - 4\,A\,C = 0 - 4\,y = -4\,y.$$

The equation is hyperbolic in the region (x, y) where $\mathrm{B}^2 - 4A\,C > 0$.

$$i,e\,,\; B^2 - A\,C > -4\,y > 0 \;\; or \;\; y > 0.$$

It is a parabolic in the region $y > 0$.

21. What is the classification of $\boldsymbol{f_x - f_{yy} = 0}$?

Solution : Here $A = 0,\; B = 0,\; C = -1$

$$\mathrm{B}^2 - 4A\,C = 0 - 4 \times 0 \times (-1)$$

So the equation is parabolic.

22. Give an example of parabolic equation.

Solution : The one dimensional heat equation $\frac{\partial \mathrm{u}}{\partial \mathrm{t}} = \propto^2 \frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2}$ is parabolic.

23. State Schmidt's explicit formula for solving heat flow equation.

Solution :

$$\mathrm{u}_{\mathrm{i,j+1}} = \lambda\,\mathrm{u}_{\mathrm{i+1,j}} + (1 - 2\lambda)\mathrm{u}_{\mathrm{i,j}} + \lambda \mathrm{u}_{\mathrm{i-1,j}}$$

$$\text{If } \lambda = \frac{1}{2}, \quad \mathrm{u}_{\mathrm{i,j+1}} = \frac{1}{2}\left[\mathrm{u}_{\mathrm{i+1,j}} + \mathrm{u}_{\mathrm{i-1,j}}\right]$$

24. Fill in the blank.

Bender Schmidt's recurrence formula is useful to solve …………………… equation.

Solution : One dimensional heat equation.

25. Write an explicit formula to solve numerically the heat equation (parabolic equation)

$u_{xx} - a\, u_t = 0$.

Solution : $u_{i,j+1} = \lambda\, u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$

Where $\lambda = \frac{k}{h^2\, a}$ (h is the space for the variable x and k is the space in the time direction).

26. What is the value of k to solve $\frac{\partial u}{\partial t} = \frac{1}{2}\, u_{xx}$ by Bender Schmidt's method with $h = 1$ if

h *and* t respectively ?

Solution :

Given $u_{xx} = 2\, \frac{\partial u}{\partial t}$ Here $a = 2,\ h = 1$

$k = \frac{ah^2}{2} = 2\frac{(1)^2}{2} = 1$ $k = 1$.

27. What is the classification of one dimensional flow equation.

Solution :

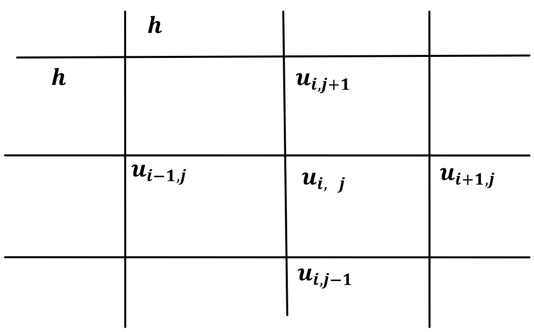
Here $A = 1,\ B = 0,\ C = 0$ $\therefore B^2 - 4\, A\, C = 0.$

Hence the one dimensional heat flow equation is parabolic.

28. Write the diagonal five point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$.

Solution :

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}\right]$$



29. Write down the standard five point formula to solve the Laplace equation $u_{xx} + u_{yy} = 0$.

Solution :

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

30. Write the difference scheme for solving the Laplace equation.

Solution : The five point difference formula for $\nabla^2\varphi = 0$ is

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right]$$

31. What is the purpose of Leibmann's process?

Solution :

The purpose of Leibmann's process is to find the solution of the Laplace equation $u_{xx} + u_{yy} = 0$ by iteration.

32. **If u satisfies the Laplace equation and $u = 100$ on the boundary of a square what will be the value of u at an interior gird point.**

Solution :

Since u satisfies Laplace equation and $u = 100$ on the boundary square.

$$u_{i,j} = \frac{1}{4}\left[100 + 100 + 100 + 100\right]$$

$$u_{i,j} = 100.$$

33. **Write the Laplace equations $u_{xx} + u_{yy} = 0$ in difference quotients.**

Solution :

$$\frac{u_{i-1,j} - 2\,u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2\,u_{i,j} + u_{i,j+1}}{k^2} = 0$$

34. **State Leibmann's iterative formula.**

Solution :

$$u_{i,j}^{n+1} = \frac{1}{4}\left[u_{i-1,j}^{n+1} + u_{i+1,j}^{n} n + u_{i,j-1}^{n} + u_{i,j+1}^{n+1}\right]$$

Part – B

1. Using Taylor series method find y at $x = 0.1\ if\ \frac{dy}{dx} = x^2y - 1,\ y(0) = 1.$

2. Solve $y' = x + y;\ \ y(0) = 1$ by Taylor's series method. Find the vlues $y\ at\ \ x = 0.1\ \ and\ \ x = 0.2.$

3. Solve $\frac{dy}{dx} = y^2 + x^2$ with $y(0) = 1.$ Use Taylor's series at $x = 0.2\ \ and\ \ 0.4.$ Find $x = 0.1$

4. Using Taylor series method find $y\ \ at\ \ x = 0.1$ correct to four decimal places from $\frac{dy}{dx} = x^2 - y\ ;\ \ y(0) = 1,\ \ with\ \ h = 0.1.$ Compute terms up to x^4.

5. Using Taylor series method, compute $y(0.2)$ & $y(0.4)$ correct to four decimal places given $\frac{dy}{dx} = 1 - 2\,x\,y\ \ \ and\ \ y(0) = 0.$

6. Using Euler's method find $y(0.2)$ & $y(0.4)$ from $\frac{dy}{dx} = x + y,\ \ y(0) = 1\ \ \ with\ \ h = 0.2.$

7. Using Euler's method solve $y' = x + y + x\,y,\ \ y(0) = 1$ compute $y\ \ at\ \ x = 0.1,$ by taking $h = 0.05.$

8. Using Euler's method find $y(0.3)$ of $y(x)$ satisfies the initial value problem. $\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)y^2,\quad y(0.2) = 1.1114.$

9. Solve $y' = 1 - y,\ \ y(0) = 0$ by modified Euler's method.

10. Using modified Euler's method, find $y(0.1)$ if $\frac{dy}{dx} = x^2 + y^2,\ y(0) = 1$.

11. Given $\frac{dy}{dx} = x^3 + y,\ y(0) = 2$. Compute $y(0.2), y(0.4)$ & $y(0.6)$ by Runge-Kutta method of fourth order.

12. Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0) = 1\ at\ x = 0.2.$

13. Using Runge-Kutta method of fourth order, find $y(0.1)\ and\ y(0.2)$ for the initial value problem $\frac{dy}{dx} = x + y^2,\ \ y(0) = 1.$

14. Apply fourth order Runge-Kutta method to determine $y(0.2)$ with $h = 0.1$ from $\frac{dy}{dx} = x^2 + y^2,\ \ y(0) = 1.$

15. Find $y(0.8)$ given that $y' = y - x^2,\ y(0.6) = 1.7379$ by using Runge-Kutta method of fourth order. Take $h = 0.1$

16. Determine the value of $y(0.4)$ using Milne's method given $y' = x\,y + y^2,\ y(0) = 1;$ use Taylor series method to get the values of $y(0.1),\ y(0.2)\ and\ y(0.3)$.

17. Using Milne's method find $y(0.2)$ if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x + y),\ given$ $y(0) = 2,\ y(0.5) = 2.636,\ \ y(1) = 3.595\ \ \&\ \ y(1.5) = 4.968$

18. Solve $y' = x - y^2,\ \ 0 \le x \le 1,\ \ y(0) = 0,\ \ y(0.2) = 0.02,\ \ y(0.4) = 0.0795,$ $y(0.6) = 0.1762$ by Milne's method to find $y(0.8)$ & $y(1)$.

19. Using Milne's method find $y(4.4)\ given\ \ 5\,x\,y' + y^2 - 2 = 0\ \ given\ \ y(4) = 1,\ y(4.1) = 1.0049,\ \ y(4.2) = 1.0097\ \ \&\ \ y(4.3) = 1.0143$

20. Given $\frac{dy}{dx} = x^3 + y,\ y(0) = 2$. The values of $y(0.2) = 2.073,\ y(0.4) = 2.452\ \ \&\ \ y(0.6) = 3.023$ are got by R-K method of fourth order. Find $y(0.8)$ by Milne's Predictor corrector method taking $h = 0.2$

21. Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1,\ t \ge 0$ with $u(x,0) = x(1-x),\ 0 < x < 1\ and$ $u(0,t) = u(1,t) = 0,\ \forall\ t > 0$ using explicit method with $\Delta x = 0.2$ for 3 time steps.

22. Solve $u_{xx} = 32\,u_t$, taking $h = 0.25$ for $t > 0\,, 0 < x < 1\ \ and\ \ u(x,0) = 0,\ u(0,t) = 0,$ $u(1,t) = t.$

23. Solve $u_t = u_{xx}$, subject $u(0,t) = 0,\ \ u(1,t) = 0\ \ \&\ \ u(x,0) = \sin \pi\,x\,,\ \ 0 < x < 1.$

24. Solve $y_{tt} = y_{xx}\ upto\ t = 0.5$ with a spacing of 0.1 subject to $y(0,t) = 0, y(1,t) = 0, y_t(x,0) = 0\ and\ \ y(x,0) = 10 + x(1-x).$

25. Approximate the solution to the wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, $0 < x < 1, t > 0, u(0,t) = u(1,t) = 0,$ $t > 0, u(x,0) = \sin 2\pi x,\ \ 0 \le x \le 1\ \ and\ \ \frac{\partial u}{\partial t}(x,0) = 0,\ \ 0 \le x \le 1\ \ with\ \ \Delta t = 0.25$ for 3 time steps.

26.Solve $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, t > 0, given\ \ u(x,0) = u_t(x,0) = u(0,t) = 0\ \ and\ \ u(1,t) =$ $100 \sin \pi t$. Compute u for four time steps with $h = 0.25$.

27. Solve $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},\ \ 0 < x < 1,\ \ t > 0,\ \ given\ \ u(x,0) = 100\,(x - x^2),\ \frac{\partial u}{\partial t}(x,0) = 0,$ $u(0,t) = u(1,t) = 0,\ \ t > 0$ by finite difference method for one time step method with $h = 0.25$.

28. Obtain a finite difference scheme to solve the Laplace equation. Solve $\Delta^2 u = 0$ at the pivotal points in the square shown fitted with square mesh. Use Leibmann's iteration procedure. (5 iteration only).

Figure