Random Experiment: An experiment is said to be a random experiment, if it's out-come can't be predicted with certainty.

Example:

If a coin is tossed, we can't say, whether head or tail will appear. So it is a random experiment.

Sample Space: The set of all possible out-comes of an experiment is called the sample space. It is denoted by $'S'$ and its number of elements are $n(s)$.

Example:

In throwing a dice, the number that appears at top is any one of 1,2,3,4,5,6.

S ={1,2,3,4,5,6} and n(s) = 6

Similarly in the case of a coin, S={Head,Tail} or {H,T} and n(s)=2.

The elements of the sample space are called sample point or event point.

Event: Every subset of a sample space is an event. It is denoted by 'E'.

Example:

In throwing a dice $S = \{1,2,3,4,5,6\}$, the appearance of an event number will be the event $E = \{2,4,6\}$.

Clearly E is a sub set of S.

Simple event: An event, consisting of a single sample point is called a simple event.

Example:

In throwing a dice, $S = \{1,2,3,4,5,6\}$, so each of $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\ and\ \{6\}$ are simple events.

Compound event: A subset of the sample space, which has more than one element is called a mixed event or compound event.

Example:

In throwing a dice, the event of appearing of odd numbers is a compound event, because $E = \{1,3,5\}$ which has '3' elements.

Equally likely events: The outcomes of an experiment are equally likely to occur when the probability of each outcome is equal.

Example:

1. In the experiment of tossing a coin:

Where A : the event of getting a "HEAD" and

B : the event of getting a "TAIL"

Events "A" and "B" are said to be equally likely events
[Both the events have the same chance of occurrence].

2. In the experiment of throwing a die:

Where A : the event of getting 1

B : the event of getting 2

...

...

F : the event of getting 6

Events "A", "B", "C", "D", "E", "F" are said to be equally likely events
[All these events have the same chance of occurrence.]

3. In the experiment of selecting integers

Where

M : the event of getting an even number

N : the event of getting an odd number

The two compound events "M" and "N" are said to be equally likely.

4. In the experiment of selecting integers
Where

P : the event of getting an odd number {1, 3, 5}

Q : the event of getting 6

The two events "P" and "Q" cannot be said to be equally likely.

Event "P" occurs when any of the elementary events of getting "1", "3" and "5" occur

Event "Q" occurs only when the elementary event of getting "6" occur.

Event "P" is three times more likely to occur than "Q"
$\Rightarrow$ "P" and "Q" are not equally likely.

Exhaustive events: One or more events are said to be exhaustive if all the possible elementary events under the experiment are covered by the event(s) considered together. In other words, the events are said to be exhaustive when they are such that at least one of the events compulsorily occurs.

Exhaustive events may be elementary or compound events. They may be equally likely or not equally likely.

Example

1. In the experiment of tossing a coin:

Where

A : the event of getting a HEAD

B : the event of getting a TAIL

The two events "A" and "B" are called exhaustive events. [When we conduct the experiment, at least one of these will occur.]

2. In the experiment of throwing a die:

Where

A : the event of getting 1

B : the event of getting 2

...

...

F : the event of getting 6

The six Events "A", "B", "C", "D", "E", "F" together are called exhaustive events. [One of these events will occur whenever the experiment is conducted.]

Classical definition of probability:

If 'S' be the sample space, then the probability of occurrence of an event 'E' is defined as:

$$P(E) = n(E)/N(S) = \frac{\text{number of elements in 'E'}}{\text{number of elements in sample space 'S'}}$$

Example:

Find the probability of getting a tail in tossing of a coin.

Solution:

Sample space $S = \{H, T\}$ and $n(s) = 2$

Event $'E' = \{T\}$ and $n(E) = 1$

$$P(E) = \frac{n(E)}{n(S)} = \frac{1}{2}$$

Sure event: Let 'S' be a sample space. If E is a subset of or equal to S then E is called a sure event.

Example:

In a throw of a dice, S={1,2,3,4,5,6}

Let E_1=Event of getting a number less than '7'.

So 'E$_1$' is a sure event.

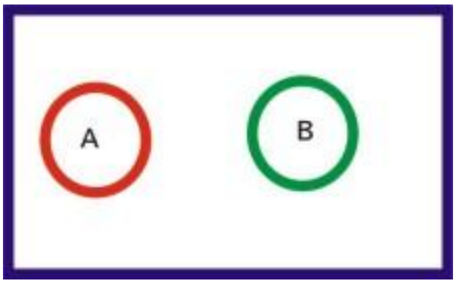
So, we can say, in a sure event n(E) = n(S)

Mutually exclusive or disjoint event: If two or more events can't occur simultaneously, that is no two of them can occur together.

So the event ‘A’ and ‘B’ are mutually exclusive if

$$A \cap B = \emptyset \ , \quad P(A \cap B) = 0$$

Pictorial Representation:



$A \cap B = \emptyset$

Example:

When a coin is tossed, the event of occurrence of a head and the event of occurrence of a tail are mutually exclusive events.

Independent or mutually independent events: Two or more events are said to be independent if occurrence or non-occurrence of any of them does not affect the probability of occurrence or non-occurrence of the other event.

If A and B are independent events then $P(A \cap B) = P(A)P(B)$

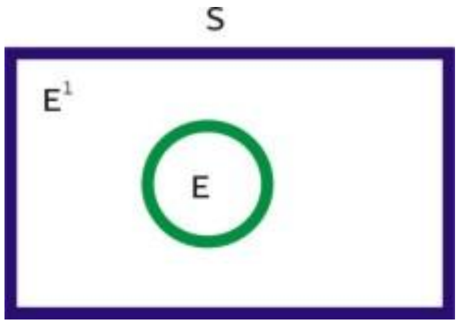
Example:

When a coin is tossed twice, the event of occurrence of head in the first throw and the event of occurrence of head in the second throw are independent events.

Difference between mutually exclusive and mutually independent events: Mutually exclusiveness is used when the events are taken from the same experiment, where as independence is used when the events are taken from different experiments.

Complement of an event: Let 'S' be the sample for random experiment, and 'E' be an event, then complement of 'E' is denoted by 'E' is denoted by E'. Here E' occurs, if and only if E' doesn't occur.

Here E’ occurs, if and only if ‘E’ doesn’t occur.



$$n(E)\ +\ n(E')\ =\ n(S)$$

Theorem 1: The probability of an event lies between 'O' and '1'.

i.e. $O \leq\ P(E)\ \leq\ 1.$

Proof: Let 'S' be the sample space and 'E' be the event.

Then

$0\ \leq\ n(E)\ \ \leq\ n\ (S)$

$$\frac{0}{n(S)} \quad \leq \quad \frac{n(E)}{n(s)} \quad \leq \quad \frac{n(S)}{n(S)}$$

or $O\ \leq P(E)\ \leq\ 1$

The number of elements in 'E' can't be less than 'O' i.e. negative and greater than the number of elements in S.

Theorem 2 : The probability of an impossible event is '0' i.e. $\boldsymbol{P}\ (\emptyset)\ =\ \mathbf{0}$

Proof: Since ∅ has no element, ⇨ $n(\emptyset)\ =\ 0$

From definition of Probability:

$$P(\emptyset)\ =\ n\ (\emptyset)\ /\ n(S)\ =\ 0\ /\ n(S)$$

⇨ $P(\emptyset)\ =\ 0$

Theorem 3 : The probability of a sure event is 1. i.e. $P(S)\ =\ 1.$ where 'S' is the sure event.

Proof : In sure event $n(E)\ =\ n(S)$

[Since Number of elements in Event 'E' will be equal to the number of element in sample-space.]

By definition of Probability :

$$P(S) \;=\; n\,(S)/\,n\,(S) \;=\; 1$$

$$\Rightarrow \quad P(S) \;=\; 1$$

Theorem 4: If two events 'A' and 'B' are such that $A \subseteq B$, then $P(A) \leq P(B)$.

Proof: $A \subseteq B$

$$\Rightarrow n(A) \;\leq\; n(B)$$

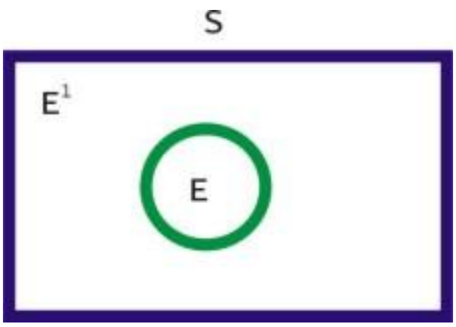
$$n(A)/\,n(S) \;\leq\; n(B)\,/\,n(S)$$

$$\Rightarrow P(A) \leq P(B)$$

Since 'A' is the sub-set of 'B", so from set theory number of elements in 'A' can't be more than number of element in 'B'.

Theorem 5 : If 'E' is any event and E' be the complement of event 'E', then $P(E) + P(E') = 1$.

Proof:



Let 'S' be the sample – space, then

$$n(E) \;+\; n(E') \;=\; n(S)$$

$$n\,(E)\,/\,n\,(S) \;+\; n\,(E')/\,n\,(S) \;=\; 1$$

$$P(E) + P(E') \;=\; 1$$

Algebra of Events: In a random experiment, let 'S' be the sample – space.

Let $A \subseteq S$ and $B \subseteq S$, where 'A' and 'B' are events.

Thus we say that :

(i) $A \cup B$, is an event occurs only when at least of 'A' and 'B' occurs. ⇨ $(A \cup B)$ means

$$(A \text{ or } B).$$

Ex.: if A = { 2,4,6,} and B = {1, 6}, than the event 'A' or 'B' occurs, if 'A' or 'B' or both occur i.e. at least one of 'A' and 'B' occurs. Clearly 'A' or 'B' occur, if the out come is any one of the outcomes 1, 2, 4, 6. That is $A \cup B = \{1,2,4,6\}$. (From set – theory).

(ii) $A \cap B$ is an event, that occurs only when each one of 'A' and 'B' occur ⇨ $(A \cap B)$ means (A and B).

Ex.: In the above example, if the out come of an experiment is '6', then events 'A' and 'B' both occur, because '6' is in both sets. That is $A \cap B = \{6\}$.

(iii) A is an event, that occurs only when 'A' doesn't occur – category of problems related to probability :

(1) Category A – When n(E) and n(S) are determined by writing down the elements of 'E' and 'S'.

(2) Category B – When n(E) and n(S) are calculated by the use of concept of permutation and combination.

(3) Category C – Problems based on $P(E) + P(E^1) = 1$

Q1: A coin is tossed successively three times. Find the probability of getting exactly one head or two heads.

Sol.: Let 'S' be the sample – space. Then,

S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

⇨ n (S) = 8

Let 'E' be the event of getting exactly one head or two heads.

Then:

$E = \{ HHT, HTH, THH, TTH, THT, HTT \}$

⇨ $n(E) = 6$

Therefore:

$P(E) = n(E)/ n(S) = 6/8 = 3/4$

Q2: Three coins are tossed. What is the probability of getting (i) all heads, (ii) two heads, (iii) at least one head, (iv) at least two heads?

Sol.: Let 'S' be the sample – space. Then

$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$

(i) Let 'E_1' = Event of getting all heads.

Then $E_1 = \{ HHH \}$

$n(E_1) = 1$

⇨ $P(E_1) = n(E_1) / n(S) = 1/8$

(ii) Let E_2 = Event of getting '2' heads.

Then:

$E_2 = \{ HHT, HTH, THH \}$

$n(E_2) = 3$

⇨ $P(E_2) = 3/8$

(iii) Let E_3 = Event of getting at least one head.

Then:

$E_3 = \{ HHH, HHT, HTH, THH, HTT, THT, TTH \}$

$n(E_3) = 7$

⇨ $P(E_3) = 7/8$

(iv) Let E_4 = Event of getting at least one head.

Then:

E_4 = { HHH, HHT, HTH, THH, }

$n(E_4)$ = 4

⇨ $P(E_4)$ = 4/8 = 1/2

Q3: What is the probability, that a number selected from 1, 2, 3, --- 2, 5, is a prime number, when each of the numbers is equally likely to be selected.

Sol.: S = { 1, 2, 3, ---- , 2, 5} ⇨ n(S) = 25

And E = { 2, 3, 5, 7, 11, 13, 17, 19, 23 } ⇨ n(E) = 9

Hence P(E) = n(E) / n(S) = 9 / 25

Q4: Two dice are thrown simultaneously. Find the probability of getting :

(i) The same number on both dice,

(ii) An even number as the sum,

(iii) A prime number as the sum,

(iv) A multiple of '3' as the sum,

(v) A total of at least 0,

(vi) A doublet of even numbers,

(vii) A multiple of '2' on one dice and a multiple of '3' on the other dice.

Sol.: Here:

S = { (1,1), (1,2) ------, (1,6), (2,1), (2,2), ---- (2,6), (3,1), (3,2), -----, (3,6), (4,1), (4,2), -------- (4,6), (5,1), (5,2), ----- (5,6), (6,1,), (6,2), --------------- (6,6) }

n(S) = 6 x 6 = 36

(i) Let E_1 = Event of getting same number on both side:

⇨ $E_1 = \{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) \}$

$n(E_1) = 6$

$P(E_1) = n(E_1)/n(S) = 6/36 = 1/6$

(ii) Let E_2 = Event of getting an even number as the sum.

$E_2 = \{ (1,1), (1,3), (1,5), (2,2), (2,4), (2,6), (3,1), (3,3), (3,5), (4,2), (4,4),$

$(4,6), (5,1), (5,5), (6,2), (6,4), (6,6) \}$

$n(E_2) = 18$ hence $P(E_2) = n(E_2)/n(S) = 18/36 = 1/2$

(iii) Let E_3 = Event of getting a prime number as the sum..

$E_3 = \{ (1,1), (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (4,1), (4,3),$

$(5,2), (5,6), (6,1), (6,5),\}$

$n(E_3) = 15$

$P(E_2) = n(E_3) / n(S) = 15/36 = 5/12$

(iv) Let E_4 = Event of getting a multiple of '3' as the sum.

$E_4 = \{ (1,2), (1,5), (2,1), (2,4), (3,3), (3,6), (4,2), (4,5), (5,1), (5,4),$

$(6,3), (6,6),\}$

$n(E_4) = 12$

$P(E_4) = n(E_4)/n(S) = 12/36 = 1/3$

(v) Let E_5 = Event of getting a total of at least 10.

$E_5 = \{ (4,6), (5,5), (5,6), (6,4), (6,5), (6,6), \}$

$n(E_5) = 6$

$P(E_5) = n(E_5)/n(S) = 6/36 = 1/6$

(vi) Let E_6 = Event of getting a doublet of even numbers.

E_6 = { (2,2), (4,4), (6,6), }

$n(E_6) = 3$

$P(E_6) = n(E_6)/ n(S) = 3/36 = 1/12$

(vii) Let E_7 = Even of getting a multiple of '2" on one dice and a multiple of '3' on the other dice.

E_7 = { (2,3), (2,6), (4,3), (4,6), (6,3), (3,2), (3,4), (3,6), (6,2), (6,4) }

$n(E_7) = 11$

$P(E_7) = n(E_7) / n(S) = 11/36$

Q5.: What is the probability, that a leap year selected at random will contain 53 Sundays?

Sol.: A leap year has 366 days, therefore 52 weeks i.e. 52 Sunday and 2 days.

The remaining 2 days may be any of the following :

(i) Sunday and Monday

(ii) Monday and Tuesday

(iii) Tuesday and Wednesday

(iv) Wednesday and Thursday

(v) Thursday and Friday

(vi) Friday and Saturday

(vii) Saturday and Sunday

For having 53 Sundays in a year, one of the remaining 2 days must be a Sunday.

$n(S) = 7$

$n(E) = 2$

$P(E) = n(E) / n(S) = 2 / 7$

CATEGORY – B

Problems based on fundamental principal of counting and permutations and combinations :

Q1. A bag contains ‘6’ red, 4 white and 8 blue balls. If three balls are drawn at random, find the probability, that

(i) ‘1’ is red and ‘2’ are white, (ii) ‘2’ are blue and 1 is red, (iii) none is red.

Sol.: We have to select ‘3’ balls, from 18 balls (6+4+8)

⇨ $n(S) = {}^{18}C_3 = 18! / (3! \times 15!) = (18 \times 17 \times 16) / (3 \times 2 \times 1) = 816$

(i) Let E_1 = Event of getting ‘1’ ball is red and ‘2’ are white

Total number of ways = $n(E_1) = {}^6C_1 \times {}^4C_2$

$= 6 \times 4 / 2$

$= 36$

$P(E_1) = n(E_1) / n(S) = 36/816 = 3/68$

(ii) Let E_2 = Event of getting ‘2’ balls are blue and ‘1’ is red.

= Total no. of ways ⇨ $n(E_2) = {}^8C_2 \times {}^6C_1$

$= (8 \times 7) / 2 \times 6 / 1 = 168$

$P(E_2) = 168 / 816 = 7/34$

(iii) Let E_3 = Event of getting ‘3’ non – red balls. So now we have to choose all the three balls from 4 white and 8 blue balls.

Total number of ways :

$n(E_3) = {}^{12}C_3 = (12 \times 11 \times 10) / (3 \times 2 \times 1) = 220$

$P(E_3) = n(E_3) / n(S) = 220 / 816 = 55/204$

Q: A box contains 12 bulbs of which ‘4’ are defective. All bulbs took alike. Three bulbs are drawn randomly.

What is the probability that :

(i) all the ‘3’ bulbs are defective?

(ii) At least ‘2’ of the bulbs chosen are defective?

(iii) At most ‘2’ of the bulbs chosen are defective?

Sol.: We have to select ‘3’ bulbs from 12 bulbs.

⇨ $n(S) = {}^{12}C_3 = (12x11x10) / (3x2x1) = 220$

(i) Let E_1 = All the ‘3’ bulbs are defective.

⇨ All bulbs have been chosen, from ‘4’ defective bulbs.

⇨ $n(E_1) = {}^{4}C_3 = 4$

⇨ $P(E_1) = n(E_1) / n(S) = 4 /220 = 1/55$

(ii) Let E_2 = Event drawing at least 2 defective bulbs. So here, we can get ‘2’ defective and 1 non-defective bulbs or 3 defective bulbs.

$n(E_2) = {}^{4}C_2 \text{ x } {}^{8}C_1 + {}^{4}C_3$ [Non-defective bulbs = 8]

$= 4x3 / 2 \text{ x } 8/1 + 4/1 = 48+4$

$n(E_2) = 52$

⇨ $P(E_2) = n(E_2) / n(S) = 52/220 = 13/55$

(iii) Let E_3 = Event of drawing at most ‘2’ defective bulbs. So here, we can get no defective bulbs or 1 is defective and ‘2’ is non-defective or ‘2’ defective bulbs.

$n(E_3) = {}^{8}C_3 + {}^{4}C_1 \text{ x } {}^{8}C_2 + {}^{4}C_2 \text{ x } {}^{8}C_1$

$= (8x7x6) / (3x2x1) + 4 \text{ x } (8x7)/2 + (4 \text{ x } 3) / 2 + 8/1$

$= 216$

$P(E_3) = n(E_3) / n(S) = 216 / 220 = 54 / 55$

Q: In a lottery of 50 tickets numbered from ‘1’ to ‘50’ two tickets are drawn simultaneously. Find the probability that:

(i) Both the tickets drawn have prime number on them,

(ii) None of the tickets drawn have a prime number on it.

Sol.: We want to select ‘2’ tickets from 50 tickets.

⇨ $n(S) = {}^{50}C_2 = (50x49) / 2 = 1225$

(i) Let E_1 = Event that both the tickets have prime numbers Prime numbers between ‘1’ to ‘50’ are :

2,3,5,7,11,13,17,19,23,29,31,37,41,43,47.

Total Numbers = 15.

We have to select ‘2’ numbers from these 15 numbers.

⇨ $n(E_1) = {}^{15}C_2$ = 15? / (2? x 13?) = (15x47) / 2 = 105

⇨ $P(E_1) = n(E_1) / n(S)$ = 105/1225 = 21/245

(ii) Non prime numbers between ‘1’ to ‘50’ = 50-15 = 35

Let E_2 = Event that both the tickets have non-prime numbers.

Now we have to select ‘2’ numbers, from ‘35’ numbers.

⇨ $n(E_2) = {}^{35}C_2$ = 35? / (2? x 33?) = (35x34) / 2 = 595

⇨ $P(E_2) = n(E_2) / n(S)$ = 595 / 1225 = 17/35

Q.: A bag contains 30 tickets, numbered from ‘1’ to ‘30’. Five tickets are drawn at random and arranged in ascending order. Find the probability that the third number is 20.

Sol.: Total number of ways of selecting ‘5’ tickets from 30 tickets = ${}^{30}C_5$

⇨ n(S) = ${}^{30}C_5$ = 30? / (5? x 25?) = (30 x 29 x 28 x 27 x 26) / (5 x 4 x 3 x 2 x 1)

n(S) = 29 x 27 x 26 x 7

Suppose the ‘5’ tickets are a1, a2,20, a4, a5

They are arranged in ascending order.

⇨ a1, a2 ⊆ {1, 2, 3, ------- , 19} and a4, a5 ⊆ { 21, 22, 23, -----, 30}

We have to select ‘2’ tickets from first ‘19’ tickets and ‘2’ tickets from last 10 tickets.

⇨ n(E) = ${}^{19}C_2$ x ${}^{10}C_2$

= 19? / (2? x 17?) = 10? / (2? x 8?) = (19 x 18) / 2 = (10 x 9) / 2

= 19 x 9 x 5 x 9

⇨ P(E) = n(E) / n(S) = (19x9x5x9) / (29x27x26x7) = 285 / 5278

Odds is Favour and Odds against an Event:

Let ‘S” be the sample space and ‘E’ be an event. Let ‘E’ devotes the complement of event ‘E’, then.

(i) Odds in favour of event ‘E’ = n(E) / n(E^1)

(ii) Odds in against of an event ‘E’ = n(E^1) / n(E)

Note : Odds in favour of ‘E’ = n(E) / n(E^1)

= [n(E) / n(S)] / [n(E^1) / n(S)] = P(E) / P(E^1)

Similarly odds in against of ‘E’ = P(E^1) / P(E)

Ex.: The odds in favours of an event are 3:5 find the probability of the occurrence of this event.

Sol.: Let ‘E’ be an event.

Then odds in favour of E = n(E) / n(E^1) = 3 / 5

$\Rightarrow$ n(E) = 3 and n(E^1) = 5

Total number of out-comes n(S) = n(E) + n(E^1) = 3+5 = 8

P(E) = n(E) / n(S) = 3 / 8

Q.: If ‘12’ persons are seated at a round table, what is the probability that two particulars persons sit together?

Sol.: We have to arrange 12 persons along a round table.

So if ‘S” be the sample – space, then n(S) = (12-1)? = 11?

n(S) = 11?

Now we have to arrange the persons in away, such that ‘2’ particulars person sit together.

Regarding that 2 persons as one person, we have to arrange 11 persons.

Total no. of ways = (11-1)? = 10? ways.

That ‘2’ persons can be arranged among themselves in 2? ways.

So, total no. of ways, of arranging 12 persons, along a round table, so that two particular person sit together : = 10? x 2?

$\Rightarrow$ n(E) = 10? x 2?

$\Rightarrow$ P(E) = n(E) / n(S) = (10? x 2?) / 11? = 2 / 11

Q.: 6 boys and 6 girls sit in a row randomly, find the probability that all the ‘6’ girls sit together.

Sol.: We have to arrange ‘6’ boys and ‘6’ girls in a row.

⇨ n(S) = 12?

Now, we have to arrange ‘6’ girls in a way, such that all of them should sit together.

Regarding all the 6 girls as one person, we have to arrange 7 person in a row.

⇨ Total no. of ways = 7?

But 6 girls can be arranged among themselves in 6? ways.

⇨ n(E) = 7? x 6?

⇨ P(E) = n(E) / n(S) = (7? x 6?) / 12? = (6x5x4x3x2x1) / (12x11x10x9x8)

P(E) = 1 / 132

Q: If from a pack of ‘52’ playing cards one card is drawn at random, what is the probability that it is either a kind or a queen?

Sol.: n(S) = Total number of ways of selecting 1 card out of 52 cards.

= ${}^{52}C_1$ = 52

n(E) = Total number of selections of a card, which is either a kind or a queen.

= ${}^{4}C_1$ + ${}^{4}C_1$ = 4 + 4 = 8

P(E) = n(E) / n(S) = 8 / 52 = 2 / 3

Q.: From a pack of 52 playing cards, three cards are drawn at random. Find the probability of drawing a king, a queen and a jack.

Sol.: Here n(S) = ${}^{52}C_3$ = 52? / (3? x 49?) = (52x51x50) / (3x2x1)

= 52x17x25

n(E) = ${}^{4}C_1$. ${}^{4}C_1$. ${}^{4}C_1$

= 4? / (1? x 3?) x 4? / (1? x 3?) = 4? / (1? x 3?)

n(E) = 4 x 4 x 4

⇨ P(E) = n(E) / n(S) = (4x4x4) / (52x17x25) = 16 / 5525

CATEGORY – C

Problems based on finding P(E’), by the use of P(E’) = 1 – P(E) :

Note : When an event has a lot of out comes, then we use this concept.

Ex.: What is the probability of getting a total of less than '12' in the throw of two dice?

Sol.: Here n(S) = 6x6 = 36

It is very difficult to find out all the cares, in which we can find the total less then '12'.

So let E = The event, that the sum of numbers is '12'.

Then E = { 6, 6}

n(E) = 1

⇨ P(E) = n(E) / n(S) = 1/36

Required probability, P(E') = 1-P(E)

= 1 – 1/36

P(E1) = 35 /36

Ex.: There are '4' envelopes corresponding to '4' letters. If the letters are placed in the envelopes at random, what is the probability that all the letters are not placed in the right envelopes?

Sol.: We have to place '4' letters in 4 envelopes.

⇨ n(S) = 4!

Now:

Let E = The event, that all the 4 letters are placed in the corresponding envelopes.

So E' = The event that all the '4' letters are not placed in the right envelope.

Here n(E) = 1

P(E) = n(E) / n(S) = 1 / 4! = 1 / 24

Required probability, P(E') = 1- P(E)

= 1 – (1/24)

P(E') = 23 / 24

Some information's about playing cards:

(1) A pack of 52 playing cards has 4 suits :

(a) Spades, (b) Hearts, (c) Diamonds, (d) Clubs.

(2) Spades and clubs are black and Hearts and Diamonds are red faced cards.

(3) The aces, kings, queens, and jacks are called face cards or honours – cards.

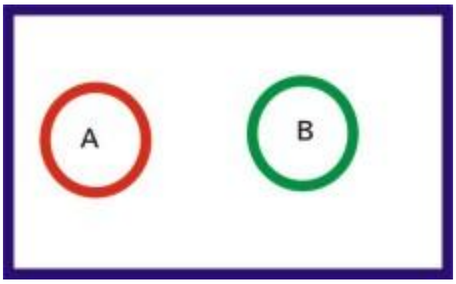
Part – 2 : (Total Probability)

Theorem – 1 : If 'A' and 'B' are mutually exclusive events then $P(A \cap B) = 0$ or P (A and B) = 0

Proof : If 'A' and 'B' are mutually exclusive events then $A \cap B = \emptyset$

⇨ $P(A \cap B) = P(\emptyset)$

$= n(\emptyset) / n(S)$ [By definition of probability]



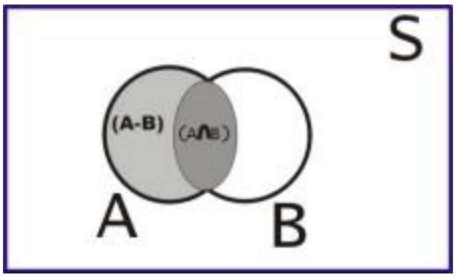
$= o / n(S)$ [Since the number of elements in a null – set is '0']

$P(A \cap B) = 0$

(2) Addition Theorem of Probability : If 'A' and 'B' by any two events, then the probability of occurrence of at least one of the events 'A' and 'B' is given by:

P(A or B) = P(A) + P(B) – P (A and B)

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$



From set theory, we have :

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

Dividing both sides by n(S) :

$n(A \cup B) / n(S) = n(A) / n(S) + n(B) / n(S) - n(A \cap B) / n(S)$

or $P(A \cup B) = p(A) + P(B) – P(A \cap B)$

Corollary : If ‘A’ and ‘B’ are mutually exclusive events,

Then $P(A \cap B) = 0$. [As we have proved]

In this case :

⇨ $P(A \cup B) = p(A) + P(B)$

Addition theorem for ‘3’ events ‘A’, ‘B’ and ‘C’ :

$P(A \cup B \cup C) = P(A) + P(B) - P(A \cap B) – P(B \cap C) – P(A \cap C) + P(A \cap B \cap C)$

Proof : $P(A \cup B \cup C) = P[(A \cup B) \cup C]$

$= P(A \cup b) + P(C) – P[(A \cup B) \cap C]$ [By addition theorem for two events]

$= P(A \cup B) - P(C) – [P(A \cap C) + P(B \cap C) – P(A \cap C \cap B \cap C)]$

$= P(A) + P(B) – P(A \cap B) + P(C) – P(A \cap C) – P(B \cap C) + P(A \cap B \cap C)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) – P(A \cap B) – P(B \cap C) – P(A \cap C) + P(A \cap B \cap C)$

Corollary : If ‘A’, ‘B’ and ‘C’ are mutually exclusive events, then $P(A \cap B) = 0$, $P(B \cap C) = 0$, $P(A \cap C) = 0$ and $P(A \cap B \cap C) = 0$.

In this case :

⇨ $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

General Form of Addition Theorem of Probability:

$$P(A_1 \cup A_2 \cup \text{-----} \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j)$$

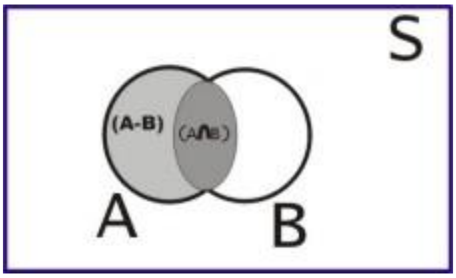
$$+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \text{-----} (-1)^{n-1} P(A_1 \cap A_2 \cap \text{----} \cap A_n)$$

Corollary : For any number of mutually exclusive events, A_1, A_2, ------ , A_n :

$P(A_1 \cup A_2 \cup \text{----} \cup A_n) = P(A_1) + P(A_2) + \text{------} + P(A_n)$

Theorem – 3 : For any two events ‘A’ and ‘B’

$P(A-B) = P(A) – P(A \cap B) = P(A \cap B^1)$



From the figure:

$(A-B) \cap (A \cap B) =$ ---------------> (i)

and

$(A-B) \cup (A \cap B) = A$

$P[(A-B) \cup (A \cap B)] = P(A)$

or $P(A-B) + P(A \cap B) = P(A)$

[From (i) $(A-B) \cap (A \cap B) =$ i.e. These events are mutually exclusive]

$\Rightarrow$ $P(A-B) = P(A) – P(A \cap B)$

or $P(A \cap B) = P(A) – P(A \cap B)$

Similarly $P(A \cap B) = P(B) – P(A \cap B)$

Proof of $P(E) + P(E^1) = 1$, by the addition theorem of probability:

We know that :

$P(A \cup B) = P(A) + P(B) – P(A \cap B)$

Putting $A = E$ and $B = E^1$

$P(E \cup E^1) = P(E) + P(E^1) – P(E \cap E^1)$ ----------------> (1)

From set theory : $E \cup E^1 = S$

And $E \cap E^1 =$

From:

$P(S) = P(E) + P(E^1) – P(\quad)$

$\Rightarrow$ $1 = P(E) + P(E^1) – 0$

or $P(E) + P(E^1) = 1$

EXAMPLES

Problems based on addition theorem of probability:

Working rule :

(i) $A \cup B$ denotes the event of occurrence of at least one of the event 'A' or 'B'

(ii) $A \cap B$ denotes the event of occurrence of both the events 'A' and 'B'.

(iii) $P(A \cup B)$ or $P(A+B)$ denotes the probability of occurrence of at least one of the event 'A' or 'B'.

(iv) $P(\cap B)$ or $P(AB)$ denotes the probability of occurrence of both the event 'A' and 'B'.

Ex.: The probability that a contractor will get a contract is '2/3' and the probability that he will get on other contract is 5/9 . If the probability of getting at least one contract is 4/5, what is the probability that he will get both the contracts ?

Sol.: Here $P(A) = 2/3$, $P(B) = 5/9$

$P(A \cup b) = 4/5$, $(P(A \cap B) = ?$

By addition theorem of Probability:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 4/5 = 2/3 + 5/9 - P(A \cap B)$

or $4/5 = 11/9 – P(A \cap B)$

or $P(A \cap B) = 11/9 – 4/5 = (55-36) / 45$

$P(A \cap B) = 19/45$

Ex2.: Two cards are drawn at random. Find the probability that both the cards are of red colour or they are queen.

Sol.: Let S = Sample – space.

A = The event that the two cards drawn are red.

B = The event that the two cards drawn are queen.

⇨ $A \cap B$ = The event that the two cards drawn are queen of red colour.

⇨ $n(S) = {}^{52}C_2$, $n(A) = {}^{26}C_2$, $n(B) = {}^{4}C_2$

$n(A \cap B) = {}^{2}C_2$

⇨ $P(A) = n(A) / n(S) = {}^{26}C_2 / {}^{52}C_2$, $P(B) = n(B) / n(S) = {}^{4}C_2 / {}^{52}C_2$

$P(A \cap B) = n(A \cap B) / n(S) = {}^{2}C_2 / {}^{52}C_2$

$P(A \cup B) = ?$

We have $P(A \cup B) = P(A) + P(B) – P(A \cap B)$

$= {}^{26}C_2 / {}^{52}C_2 + {}^{4}C_2 / {}^{52}C_2 – {}^{2}C_2 / {}^{52}C_2$

$= ({}^{26}C_2 + {}^{4}C_2 – {}^{2}C_2) / {}^{52}C_2$

= (13X25+2X3-1) / (26X51)

$P(A \cup B) = 55/221$

Ex.3: A bag contains '6' white and '4' red balls. Two balls are drawn at random. What is the chance, they will be of the same colour?

So.: Let S = Sample space

A = the event of drawing '2' white balls.

B = the event of drawing '2' red balls.

$A \cup B$ = The event of drawing 2 white balls or 2 red balls.

i.e. the event of drawing '2' balls of same colour.

⇨ $n(S) = {}^{10}C_2 = 45$

$n(A) = {}^{6}C_2 = (6 \times 5) / 2 = 15$

$n(B) = {}^{4}C_2 = (4 \times 3) / 2 = 6$

$P(A) = n(A) / n(S) = 15/45 = 1/3$

$P(B) = n(B) / n(S) = 6/45 = 2/15$

⇨ $P(A \cup B) = P(A) + P(B)$

$= 1/3 + 2/15 = (5+2) / 15$

$P(A \cup B) = 7/15$

Ex.: For a post three persons 'A', 'B' and 'C' appear in the interview. The probability of 'A' being selected is twice that of 'B' and the probability of 'B' being selected is thrice that of 'C', what are the individual probability of A, B, C being selected?

Sol.: Let ‘E_1’, ‘E_2’, ‘E_3’ be the events of selections of A, B, and C respectively.

Let $P(E_3) = x$

⇨ $P(E_2) = 3. P(E_3) = 3x$

and $P(E_1) = 2P(E_2) = 2 \times 3x = 6x$

As there are only ‘3’ candidates ‘A’, ‘B’ and ‘C’ we have to select at least one of the candidates A or B or C, surely.

⇨ $P(E_1 \cup E_2 \cup E_3) = 1$

and E_1, E_2, E_3 are mutually exclusive.

⇨ $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$

$1 = 6x + 3x + x$

⇨ $10x – 1$ or $x = 1/10$

⇨ $P(E_3) = 1/10$, $P(E_2) = 3/10$ and $P(E_1) = 6/10 = 3/5$

Conditional Probability:

The conditional probability of an event B in relationship to an event A is the probability that event B occurs given that event A has already occurred. The notation for conditional probability is P(B|A), read as *the probability of B given A*. The formula for conditional probability is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example 1: A jar contains black and white marbles. Two marbles are chosen without replacement. The probability of selecting a black marble and then a white marble is 0.34, and the probability of selecting a black marble on the first draw is 0.47. What is the probability of selecting a white marble on the second draw, given that the first marble drawn was black?

Solution:

$$\text{P(White|Black)} = \frac{\text{P(Black and White)}}{\text{P(Black)}} = \frac{0.34}{0.47} = 0.72 = 72\%$$

Example 2: The probability that it is Friday and that a student is absent is 0.03. Since there are 5 school days in a week, the probability that it is Friday is 0.2. What is the probability that a student is absent given that today is Friday?

Solution:

$$\text{P(Absent|Friday)} = \frac{\text{P(Friday and Absent)}}{\text{P(Friday)}} = \frac{0.03}{0.2} = 0.15 = 15\%$$

Example 3: At Kennedy Middle School, the probability that a student takes Technology and Spanish is 0.087. The probability that a student takes Technology is 0.68. What is the probability that a student takes Spanish given that the student is taking

Technology?

Solution: P(Spanish|Technology) = $\frac{\text{P(Technology and Spanish)}}{\text{P(Technology)}}$ = $\frac{0.087}{0.68}$ = 0.13 = 13%

Total Probability theorem:

Let $A_1, A_2, \dots, A_n$ are mutually exclusive events whose probabilities sum to unity and B be any arbitrary event, then

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_n)P(A_n)$$

Example:

One bag contains 5 white and 4 black balls. Another bag contains 7 white and 9 black balls. A ball is transferred from the first bag to the second and then a ball is drawn from second. What will be the probability that the ball is white?

Solution:

Let A_1 denotes the event that the first ball chosen is white, A_2 denotes the event that the first ball chosen is black, and B denotes the event the the ball from the second bag is white.

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) = \frac{5}{9}\frac{8}{17} + \frac{4}{9}\frac{7}{17} = \frac{68}{153}$$

Example: There are three boxes, each containing a different number of light bulbs. The first box has 10 bulbs, of which four are dead, the second has six bulbs, of which one is dead, and the third box has eight bulbs of which three are dead. What is the probability of a dead bulb being selected when a bulb is chosen at random from one of the three boxes?

Solution:

Let A_1, A_2, A_3 denotes the events of selecting bulbs from bags 1,2 and 3 respectively. Let B denotes the event the bulb selected are dead.

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$$

$$P(B|A_1) = \frac{4}{10}, P(B|A_2) = \frac{1}{6}, \qquad P(B|A_3) = \frac{3}{8}$$

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

$$= \frac{1}{3}\frac{4}{10} + \frac{1}{3}\frac{1}{6} + \frac{1}{3}\frac{3}{8} = \frac{113}{360}$$

Bayes theorem:

Let $A_1, A_2, \ldots, A_n$ be a set of mutually exclusive events that together form the sample space S. Let B be any event from the same sample space, such that $P(B) > 0$. Then,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{n} P(A_i)P(B|A_i)}$$

Example 1

Marie is getting married tomorrow, at an outdoor ceremony in the desert. In recent years, it has rained only 5 days each year. Unfortunately, the weatherman has predicted rain for tomorrow. When it actually rains, the weatherman correctly forecasts rain 90% of the time. When it doesn't rain, he incorrectly forecasts rain 10% of the time. What is the probability that it will rain on the day of Marie's wedding?

Solution: The sample space is defined by two mutually-exclusive events - it rains or it does not rain. Additionally, a third event occurs when the weatherman predicts rain. Notation for these events appears below.

- Event A_1. It rains on Marie's wedding.
- Event A_2. It does not rain on Marie's wedding
- Event B. The weatherman predicts rain.

In terms of probabilities, we know the following:

- $P(A_1) = 5/365 = 0.0136985$ [It rains 5 days out of the year.]
- $P(A_2) = 360/365 = 0.9863014$ [It does not rain 360 days out of the year.]
- $P(B | A_1) = 0.9$ [When it rains, the weatherman predicts rain 90% of the time.]
- $P(B | A_2) = 0.1$ [When it does not rain, the weatherman predicts rain 10% of the time.]

We want to know $P(A_1 | B)$, the probability it will rain on the day of Marie's wedding, given a forecast for rain by the weatherman. The answer can be determined from Bayes' theorem, as shown below.

$$P(A_1 | B) = \frac{P(A_1) P(B | A_1)}{P(A_1) P(B | A_1) + P(A_2) P(B | A_2)}$$

$$P(A_1 | B) = (0.014)(0.9) / [(0.014)(0.9) + (0.986)(0.1)]$$

$$P(A_1 | B) = 0.111$$

Example 2: There are 3 urns A, B and C each containing a total of 10 marbles of which 2, 4 and 8 respectively are red. A pack of cards is cut and a marble is taken from one of the urns depending on the suit shown - a black suit indicating urn A, a diamond urn B, and a heart urn C. What is the probability a red marble is drawn?

Solution:

Let $U_1, U_2 \; and \; U_3$ are the events of selecting marbles from urns $A, B \; and \; C$ respectively. Let R be the event of selecting red marble from the urns.

$$P(U_1) = P(U_2) = P(U_3) = \frac{1}{3}$$

$$P(R|U_1) = \frac{2}{10}, P(R|U_2) = \frac{4}{10}, \qquad P(R|U_3) = \frac{8}{10}$$

Probability of drawing a red marble from urn A,

$$P(U_1|R) = \frac{P(U_1)P(R|U_1)}{P(U_1)P(R|U_1) + P(U_2)P(R|U_2) + P(U_3)P(R|U_3)}$$

$$= \frac{\frac{1}{3}\frac{2}{10}}{\frac{1}{3}\frac{2}{10} + \frac{1}{3}\frac{4}{10} + \frac{1}{3}\frac{8}{10}} = \frac{2}{14} = \frac{1}{7}$$

Probability of drawing a red marble from urn B,

$$P(U_2|R) = \frac{P(U_2)P(R|U_2)}{P(U_1)P(R|U_1) + P(U_2)P(R|U_2) + P(U_3)P(R|U_3)}$$

$$= \frac{\frac{1}{3}\frac{4}{10}}{\frac{1}{3}\frac{2}{10} + \frac{1}{3}\frac{4}{10} + \frac{1}{3}\frac{8}{10}} = \frac{4}{14} = \frac{2}{7}$$

Probability of drawing a red marble from urn C,

$$P(U_3|R) = \frac{P(U_3)P(R|U_3)}{P(U_1)P(R|U_1) + P(U_2)P(R|U_2) + P(U_3)P(R|U_3)}$$

$$= \frac{\frac{1}{3}\frac{8}{10}}{\frac{1}{3}\frac{2}{10} + \frac{1}{3}\frac{4}{10} + \frac{1}{3}\frac{8}{10}} = \frac{8}{14} = \frac{4}{7}$$

Bernoulli trial:

An experiment having only two possible outcomes, usually denoted *success* and *failure,* with the properties that the probability of occurrence of each outcome is the same in each trial and the occurrence of one excludes the occurrence of the other in any given trial.

A sequence of **Bernoulli trials** occurs when a Bernoulli experiment is performed several independent times so that the probability of success, say, **p**, remains the same from trial to trial. That is, in such a sequence we let **p** denote the probability of success on each trial. In addition, frequently $q=1-p$ denote the probability of failure; that is, we shall use **q** and $1-p$ interchangeably.

Example: when rolling a die, we may be only interested whether 1 shows up, in which case,naturally, P(S) = 1/6 and P(F) = 5/6. If, when rolling two dice, we are only interested whether the sum on two dice is 11, P(S) = 1/18, P(F) = 17/18.