UNIT- II TWO DIMENSIONAL RANDOM VARIABLES

PART A (2 Marks)

1.State the basic properties of joint distribution of (X,Y) where x and Y are random variables.

$F(x,y) = P[X \le x, Y \le y]$. $F(-\infty,\infty) = 1$; $F(-\infty,y)$= Marginal distribution of Y and $F(x,\infty)$ = Marginal distribution of X

2. The joint p.d.f. f(X,Y) is $f(x,y)= 6e^{-2x-3y}$, $x \ge 0$, $y \ge 0$, find the conditional density of Y given X.

Marginal of X is $f(x) = \int_0^\infty 6e^{-2x} e^{-3y}\, dy$,

$= 2^{e-2x}$, $x>0$

Conditional of Y given x :

$f(y/x) = f(x/y) / f(x)$

$= (6e^{-2x} e^{-3y})/ 2e^{-2x}$

$= 3 e^{-3x}$, $y \ge 0$.

3. State the basic properties of joint distribution of (x,y) when X and Y are random variables.

$F(x,y) = P(X \le x, Y \le y)$ $F(\infty$
$\infty) = 1$
$F(\infty, y)$ = Marginal distribution of Y
$F(x, \infty)$ = Marginal distribution of X.
$F(-\infty, y) = 0$, $F(x, -\infty)=0$.

4. If 2 random variables have the joint density $f(x_1, x_2)= x_1x_2$, $0<x_1<1$, $0<x_2<2$. Find the probability that both random variables will take on values less than 1.

$$P(x_1 \le 1, x_2 \le 1) = \int_0^1 \int_0^1 x_1x_2 \, dx_2 dx_1.$$

$$= \int_0^1 x_1(x_2^2/2)\Big|_0^1 dx_1$$

$$= (1/2) (x_1^2/ 2)\Big|_0^1$$

$=1/4$

5. The joint probability mass function of (X,Y) is given by P(x,y) = k (2x + 3y) x = 0,1,2 y = 1,2,3. Find the marginal probability distribution of X

X \ Y	1	2	3
0	3k	6k	9k
1	5k	8k	11k
2	7k	10k	13k

$\sum \sum P(x,y) = 1 \Rightarrow 72\, k = 1 \Rightarrow k = 1/ 72$

Marginal distribution of X :

X	0	1	2
P(X = x)	18/72	24/72	30/72

6. If f(x,y) = k (1 – x – y), 0 < x,y < ½ . Find K.

$$K\int_0^{1/2}\int_0^{1/2} (1-x-y)\,dx\,dy = 1 \Rightarrow K\int_0^{1/2} \left((1-y)/2 - 1/8\right) dy = 1 \Rightarrow K\left(3y/8 - y^2/4\right)\Big|_0^{1/2} \Rightarrow k/8=1$$

$\Rightarrow K = 8$

7. If X and Y are independent random variables with variances 2 and 3.

Find Var(3X + 4Y)

$Var(3X + 4Y) = 3^2\, Var(x) + 4^2\, Var(y) = 9 * 2 + 16 * 3 = 66$

8. If X and Y are independent random variables, find covariance between X + Y and X – Y.

$Cov(X + Y, X - Y) = E[(X + Y) - E(X + Y))[(X - Y) - E(X - Y)]$

$= E[(X - E(X) + Y - E(Y))(X - E(Y) - (Y - E(Y))]$

$= E[(X - E(X))^2 - (Y - E(Y))^2$

$= Var\,X - Var\,Y.$

9. If X and Y have joint probability density function f(x,y) = x + y 0 < x,y < 1, Check whether X and Y are independent

Marginal density function of X is $f_X(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x + y)\,dy = x + ½$, $0 < x < 1$

Marginal density function of Y is $f_Y(y) = \int_0^1 f(x,y)\,dx = \int_0^1 (x + y)\,dy = y + ½$, $0 < y < 1$

$f_X(x) . f_Y(y) = (x + ½) . (y + ½) \neq$

f(x,y) X and Y are not independent.

10.Let X and Y be random variables with joint density function f (x, y) = 4xy 0 < x,y < 1. Find E(xy).

$$E(xy) = \int_0^1\int_0^1 xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy\, 4xy\,dx\,dy = 4\int_0^1\int_0^1 x^2 y^2\,dx\,dy = 4/9.$$

11.The joint pdf of two random variables X and Y is given by f(x,y) = 1/8 x (x-y), 0 < x < 2, -x < y < x . Find f (y/x).

$$f_X(x) = \int_{-x}^{x} 1/8\, x\,(x-y)\,dy = x^3/4,\ 0 < x < 2$$

$$f(y/x) = f(x,y) / f_X(x) = 1/8\, x\,(x-y) / x^3/4 = (x-y) / 2x^2,\ -x < y < x$$

12. The joint pdf of (X,Y) is $f(x,y) = 6e^{-2x-3y}$, $x \geq 0$, $y \geq 0$. find the conditional density of Y given X.

$$\text{Marginal of X is } f(x) = \int_0^\infty 6e^{-2x-3y}\,dy$$

$$= 2e^{-2x},\ x > 0.$$

Conditional of Y given X:

$$f(y/x) = f(x,y) / f(x)$$

$$= 6e^{-2x} e^{-3y} / xe^{-2x}.$$

$$= 3e^{-3y},\ y \geq 0.$$

13. State the basic properties of joint distribution of (X,Y) when X and Y are random variable.

$F(x,y) = P(X \leq x, Y \leq y)$ $F(\infty, \infty) = 1$.

$F(\infty, y)$ = Marginal distribution of Y.

$F(x,\infty)$ = Marginal distribution of X.

$F(-\infty, y) = 0$, $F(x, -\infty) = 0$.

14. If 2 random variables have the joint density $f(x_1, x_2) = x_1x$, $0 < x_1 < 1$, $0 < x_2 < 2$. Find the probability that both random variables will take on values less than 1.

$$P[x_1 \leq 1, x_2 \leq 1] = \int_0^1 \int_0^1 x_1x_2\,dx_2\,dx_1$$

$$= \int x_1(x_2 / 2)\,dx_1 = 1/4.$$

15.. The conditional p.d.f. of X and Y = y is given by $f(x / y) = [(x+y) / (1+y)] e^{-X}$, $0< x < \infty$, $0 < y < \infty$. Find $P(x < 1 /Y=2)$.

When y=2, $f(x / y=2) =[(x+2) /3] e^{-X}$

$$P(X <1 / Y=2) = \int_0^1 [(x+2) / 3] e^{-X}$$

$$= (1/3) \int_0^1 x e^{-X} dx + (2/3) \int_0^1 e^{-X} dx$$

$$= 1 - (4/3) e^{-1}$$

16. If X and Y are independent random variables find covariance between X+Y and X – Y.

$$\begin{aligned} Cor(X+Y, X-Y) &= E [\{(X+Y)-E(X+Y)\} \{(X-Y) - E(X-Y)] \\ &= E[\{ X- E(X) +Y - E(Y)\} \{ (X-E(Y)) - (Y-E(Y))\}] \\ &= E[\{X-E(X)\}^2 - \{Y - E(Y) \}^2 \\ &= Var X - Var Y. \end{aligned}$$

17. If X and Y are independent random variable with variances 2 and 3, find Var (3X +4Y).

$$\begin{aligned} Var(3X+ 4Y) &= 3^2 Var(X) + 4^2 Var(Y) \\ &= 9 X 2 + 16 X 3 \\ &= 66. \end{aligned}$$

18. If the joint pdf of (X,Y) is given by $f(x,y) = e^{-(x+y)}$, $x \ge 0, y \ge 0$ Find E (XY)

$$E (XY) = \int_0^\infty \int_0^\infty xye^{-(x+y)} dxdy = \int_0^\infty xe^{-x} dx \int_0^\infty ye^{-y} dy = 1$$

19.The regression lines between two random variables X and Y is given by 3x + 4y =10 and 3x + 4y = 12. Find the correlation between two regression lines.

$3x + 4y =10 \Rightarrow b_{yx} = -\frac{3}{4}$. $3x + 4y = 12 \Rightarrow b_{xy} = -\frac{1}{3}$

$r^2 = -\frac{3}{4} . -\frac{1}{3} = \frac{1}{4} \Rightarrow r = \frac{1}{2}$

20. **Distinguish between correlation and regression**

By correlation we mean the casual relationship between two or more variables. By regression we mean the average relationship between two or more variables.

21. Why there are two regression lines?

Regression lines express the linear relationship between two variable X and Y. Since any of them is taken as independent variable, we have two regression lines.

22. State central limit theorem?

If $X_1, X_2 \ldots X_n$ be a sequence of independent identically distributed R.Vs with $E(X_i) = \mu$ and var $(X_i) = \sigma^2$ for o = 1,2… and if $S_n = X_1 + X_2 + \ldots X_n$, then under certain general conditions, S_n follows a normal distribution with mean $n\mu$ and variance $n\sigma^2$ as n tends to ∞

23.If Y_1 and Y_2 are two independent random variables, then covariance (Y_1, Y_2) = 0. IS the converse of the above statement true? Justify your answer.

The converse not true. Consider $X \sim N(0,1)$ and $Y = X^2$ since $X \sim N(0,1)$. $E(X) = 0$; $E(X^3) = E(XY) = 0$ since all odd moments vanish. Cov (x, y) = 0 but X and Y are not independent.

24. Find the value of k, if f(x,y) = k (1- x) (1- y) for 0 < x,y < 1 is to be a joint density function.

We know $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1 \Rightarrow \int_0^1\int_0^1 k(1-x)(1-y)\,dxdy = 1$

$$k\int_0^1 \left(x - \frac{x^2}{2} - xy + \frac{x^2 y}{2} \right)_0^1 dy = 1$$

$$\Rightarrow k\int_0^1 \left(\frac{1}{2} - y + \frac{y}{2} \right) dy = 1$$

$$\Rightarrow k\left(\frac{y}{2} - \frac{y^2}{2} + \frac{y^2}{4} \right)_0^1 = 1$$

$$k\left(\frac{1}{4}\right) = 1 \Rightarrow k = 4$$

25.If X has an exponential distribution with parameter α find the pdf of Y = log X

$$f_Y(y) = f_X(x)\left|\frac{\mathrm{dx}}{\mathrm{dy}}\right| = e^y \alpha e^{-\alpha e^{-y}}, -\infty < y < \infty$$

26. State any two properties of regression coefficient.

i. Both regression coefficients cannot be greater than unity.

ii. If b_1, b_2 are positive then $(b_1 + b_2)/2$ = A.M of $b_1, b_2 > r$.

27. What are types of regression analysis?.

The regression analysis between independent and dependent variables can be classified in different ways.

i. Simple and multiple

ii. Total and partial

iii. Linear and non – linear.

28. Say true or false : A linear transformation of random variables affect the type of input probability density function.

Answer: False.

29. Say true or false : Correlation between variables gives the degree of relationship between them.

Answer: True.

30. State any two properties of Correlation coefficient.

i. Correlation coefficient is independent of change of origin and scale.

ii. Two independent variables are uncorrelated i.e if X and Y are independent , r = 0.

PART B (12 Marks)

1. If f (x, y) = x+y , 0< x <1, 0< y < 1

1. 0 , Otherwise

b. Compute the correlation co- efficient between X and Y.

2. The joint p.d.f of a two dimensional randm variable (X, Y) is given by f(x, y) = (8 /9) xy, $1 \le x \le y \le 2$ find the marginal density unctions of X and Y. Find also the conditional density function of Y / X =x, and X / Y = y.

3. The joint probability density function of X and Y is given by f(x, y) = (x + y) /3, $0 \le x \le 1$ & 0<y <2 obtain the egression of Y on X and of X on Y.

4. If the joint p.d.f. of two random variable is given by $f(x_1, x_2) = 6\, e^{-2x_1 - 3x_2}$, $x_1 > 0$, $x_2 > 0$. Find the probability that the first random variable will take on a value between 1 and 2 and the second random variable will take on avalue between 2 and 3. \also find the probability that the first random variable will take on a value less than 2 and the second random variable will take on a value greater than 2.

5. If two random variable have joint p.d.f. $f(x_1, x_2) = (2/3)(x_1 + 2x_2)$ $0 < x_1 < 1$, $0 < x_2 < 1$.

6. Find the value of k, if f(x,y) = k xy for 0 < x,y < 1 is to be a joint density function. Find P(X + Y < 1) . Are X and Y independent.

7. If two random variable has joint p.d.f. $f(x, y) = (6/5)(x + y^2)$, 0 < x < 1 , 0< y <1.Find P(0.2 < X < 0.5) and P(0.4 < Y < 0.6).

8. Two random variable X and Y have p.d.f $f(x, y) = x^2 + (xy/3)$, $0 \le x \le 1$, $0 \le y \le 2$. Prove that X and Y are not independent. Find the conditional density function.

9. X and Y are 2 random variable joint p.d.f. $f(x, y) = 4xy\, e^{-(x^2+y^2)}$, $x, y \ge 0$, find the p. d. f. of $\sqrt{x^2+y^2}$.

10. Two random variable X and Y have joint f(x y) 2 – x –y, 0< x <1, 0< y < 1. Find the Marginal probability density function of X and Y. Also find the conditional density unction and covariance between X and Y.

11. Let X and Y be two random variables each taking three values –1, 0 and 1 and having the joint p.d.f.

Y	X		
	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.1

Prove that X and Y have different expections. Also Prove that X and Y are uncorrelated and find Var X and Var Y

12. 20 dice are thrown. Find the approximate probability tat the sum obtained is between 65 and 75 using central limit theorem.

13. Examine whether the variables X and Y are independent whose joint density is $f(x, y) = x e^{-xy-x}$, $0 < x, y < \infty$

14. Let X and Y be independent standard normal random variables.
Find the pdf of $z = X / Y$.

15. Let X and Y be independent uniform random variables over (0,1) .
Find the PDF of $Z = X + Y$