The Bisection method or BOLZANO's method or Interval halving method:

Find the positive root of $x^3 - x = 1$ correct to four decimal places by bisection method.

Solution:

Let $f(x) = x^3 - x - 1$

Here $f(0) = -1 = -ve,\ f(1) = -ve, f(2) = 5 = +ve$

Hence a root lies between 1 and 2.

Here $a = 1\ and\ b = 2$

$$Here\ x_0 = \frac{a+b}{2} = \frac{1+2}{2} = 1.5, f(x_0) = f(1.5) = 0.875 = +ve$$

Since $f(a)$ is negative and $f(x_0)$ is positive

The root lies between a and x_0 i.e. 1 and 1.5, put $b = x_0$

$$x_1 = \frac{1+1.5}{2} = 1.25, f(1.25) = -0.2969 = -ve$$

The root lies between x_1 and b i.e. 1.25 and 1.5, put $a = x_1$

$$x_2 = \frac{1.25+1.5}{2} = 1.375, f(1.375) = 0.2246 = +ve$$

Therefore the root lies between a and x_2 i.e. 1.25 and 1.375

Proceeding like this and these values are put in a tabular form as follows

i	a	b	x_n	$f(x_n)$
0	1	2	1.5	0.875
1	1	1.5	1.25	-0.2969
2	1.25	1.5	1.375	0.2246
3	1.25	1.375	1.3125	-0.0515
4	1.3125	1.375	1.3438	0.0828
5	1.3125	1.3438	1.3282	0.0149
6	1.3125	1.3282	1.3204	-0.0183
7	1.3204	1.3282	1.3243	-0.0018
8	1.3243	1.3282	1.3263	0.0068
9	1.3243	1.3263	1.3253	0.0025
10	1.3243	1.3253	1.3248	0.0003
11	1.3243	1.3248	1.3246	-0.0005
12	1.3246	1.3248	1.3247	-0.0001
13	1.3247	1.3248	1.3248	0.0003
14	1.3247	1.3248	1.3248	0.0003

Since the values in 13th and 14th iteration are same, the root is 1.3248.

Find the positive root of $x - \cos x = 0$ by bisection method.

Solution:

Let $f(x) = x - \cos x$

Here $f(0) = -1 = -ve, \; f(1) = 1 - \cos 1 = 0.4597 = +ve$

Hence a root lies between 0 and 1.

Here $a = 0 \; and \; b = 1$

$$Here \; x_0 = \frac{a+b}{2} = \frac{0+1}{2} = 0.5, f(x_0) = f(0.5) = -0.3776 = -ve$$

Since $f(x_0)$ is negative and $f(b)$ is positive

The root lies between x_0 and b i.e. 0.5 and 1, put $a = x_0$

$$x_1 = \frac{0.5+1}{2} = 0.75, f(0.75) = 0.0183 = +ve$$

The root lies between a and x_1 i.e. 0.5 and 0.75, put $b = x_1$

$$x_2 = \frac{0.5+0.75}{2} = 0.625, f(0.625) = -0.186 = -ve$$

Therefore the root lies between x_2 and b i.e. 0.625 and 0.75

Proceeding like this and these values are put in a tabular form as follows

i	a	b	x_n	$f(x_n)$
0	0	1	0.5	-0.3776
1	0.5	1	0.75	0.0183
2	0.5	0.75	0.625	-0.186
3	0.625	0.75	0.6875	-0.0853
4	0.6875	0.75	0.7188	-0.0338
5	0.7188	0.75	0.7344	-0.0078
6	0.7344	0.75	0.7422	0.0052
7	0.7344	0.7422	0.7383	-0.0013
8	0.7383	0.7422	0.7403	0.002
9	0.7383	0.7403	0.7393	0.0004
10	0.7383	0.7393	0.7388	-0.0005
11	0.7388	0.7393	0.7391	0

Since the value of the last column is zero in last column of 11^{th} iteration, the root is 0.7391

Regula Falsi method or False position method:

Solve for a positive root of $x^3 - 4x + 1 = 0$ by Regula Falsi method.

Solution: Let $f(x) = x^3 - 4x + 1$

Here $f(0) = 1 = +ve, f(1) = -2 = -ve$

$\therefore$ A root lies between 0 and 1. Here $a = 0$ and $b = 1$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = -\frac{1}{-2-1} = 0.3333$$

$$f(x_1) = f(0.3333) = -0.2963 = -ve$$

Hence the root lies between 0 and 0.3333 i.e. $a = 0 \; and \; b = 0.3333$

$$(Since\ f(0) = +ve\ and\ f(0.3333) = -ve)$$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(0.3333) - 1 \times f(0)}{f(0.3333) - f(0)} = 0.2571$$

$$f(0.2571) = -0.0114 = -ve$$

Hence the root lies between 0 and 0.3333 i.e. $a = 0 \; and \; b = 0.2571$

$$(Since\ f(0) = +ve\ and\ f(0.2571) = -ve)$$

Continuing in the same way and the values are put in a table below

i	a	b	$f(a)$	$f(b)$	x_n	$f(x_n)$
1	0	1	1	-2	0.3333	-0.2962
2	0	0.3333	1	-0.2962	0.2571	-0.0114
3	0	0.2571	1	-0.0114	0.2542	-0.0004
4	0	0.2542	1	-0.0004	0.2541	0

Since the value of the last column of 4th iteration is zero, the root is 0.2541

Find a positive root of $xe^x = 2$ by the method of false position.

Solution: Let $f(x) = xe^x - 2$

Here $f(0) = -2 = -ve, f(1) = 0.7183 = +ve$

∴ A root lies between 0 and 1. Here $a = 0$ and $b = 1$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(1) - 1 \times f(0)}{f(1) - f(0)} = \frac{2}{0.7183 + 2} = 0.3333$$

$$f(x_1) = f(0.3333) = -0.2963 = -ve$$

Hence the root lies between 0 and 0.3333 i.e. $a = 0 \; and \; b = 0.3333$

$$(Since\ f(0) = +ve\ and\ f(0.3333) = -ve)$$

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0 \times f(0.3333) - 1 \times f(0)}{f(0.3333) - f(0)} = 0.2571$$

$$f(0.2571) = -0.0114 = -ve$$

Hence the root lies between 0 and 0.3333 i.e. $a = 0 \; and \; b = 0.2571$

$$(Since \; f(0) = +ve \; and \; f(0.2571) = -ve)$$

Continuing in the same way and the values are put in a table below

i	a	b	$f(a)$	$f(b)$	x_n	$f(x_n)$
1	0	1	-2	0.7183	0.7358	-0.4643
2	0.7358	1	-0.4643	0.7183	0.8395	-1.7664
3	0.8395	1	-0.0564	0.7183	0.8512	-1.7881
4	0.8512	1	-0.0061	0.7183	0.8525	-1.7904
5	0.8525	1	-0.0005	0.7183	0.8526	-1.7906
6	0.8526	1	0	0.7183	0.8526	-1.7906

Since the value of the x_n column of 5^{th} and 6^{th} iteration are same, the root is 0.8526 up to four decimal places.

Gauss Jacobi method:

Solve by Gauss Jacobi method, the system

$$12x + y + z = 31, 2\,x + 8y - z = 24, 3x + 4y + 10z = 58$$

Solution:

$$|1| + |1| < |12|, |2| + |-1| < |8|, |3| + |4| < |10|$$

Therefore the given equations are diagonally dominant, so we can use Gauss Jacobi method.

Proceeding in the same manner and writing the solutions in the tabular form we get

i	x	y	z
1	2.5833	3	5.8
2	1.85	3.0792	3.825
3	2.008	3.0156	4.0133
4	1.9976	2.9997	3.9914

5	2.0007	2.9995	4.0008
6	2	2.9999	4
7	2	3	4
8	2	3	4

Since the values in 7th and 8th iteration are same, the solution is $x = 2, y = 3, z = 4$ up to four decimal places.

Solve by Gauss Jacobi method, the system

$$9x - y + 2z = 9, x + 10y - 2z = 15, 2x - 2y - 13z = -17$$

Solution:

$$|-1| + |2| = 3 < |9|, |1| + |-2| < |10|, |2| + |-2| < |-13|$$

Therefore the given equations are diagonally dominant, so we can use Gauss Jacobi method.

$$x = \frac{1}{9}(9 + y - 2z)$$

$$y = \frac{1}{10}(15 - x + 2z)$$

$$z = \frac{1}{13}(17 + 2x - 2y)$$

1st iteration

$$x^{(1)} = \frac{1}{9}\left(9 + 0 - 2(0)\right) = 1$$

$$y^{(1)} = \frac{1}{10}\left(15 - 0 + 2(0)\right) = 1.5$$

$$z^{(1)} = \frac{1}{13}(17 + 2(0) - 2(0)) = 1.3077$$

2nd iteration

$$x^{(2)} = \frac{1}{9}\left(9 + 1.5 - 2(1.3077)\right) = 0.8761$$

$$y^{(2)} = \frac{1}{10}\big(15 - 1.5 + 2(1.3077)\big) = 1.6615$$

$$z^{(2)} = \frac{1}{13}(17 + 2(1) - 2(1.5)) = 1.2308$$

Proceeding in the same manner and writing the solutions in the tabular form we get

i	x	y	z
1	1	1.5	1.3077
2	0.8761	1.6615	1.2308
3	0.9111	1.6586	1.1869
4	0.9205	1.6463	1.1927
5	0.9179	1.6465	1.196
6	0.9172	1.6474	1.1956
7	0.9174	1.6474	1.1954
8	0.9174	1.6473	1.1954
9	0.9174	1.6473	1.1954

Since the values in 8th and 9th iteration are same, the solution is $x = 0.9174, y = 1.6473, z = 1.1954$ up to four decimal places.

Gauss Seidel method

Solve by Gauss Seidel method, the system

$$12x + y + z = 31, 2\,x + 8y - z = 24, 3x + 4y + 10z = 58$$

Solution:

$$|1| + |1| < |12|, |2| + |-1| < |8|, |3| + |4| < |10|$$

Therefore the given equations are diagonally dominant, so we can use Gauss Seidel method.

$$x = \frac{1}{12}(31 - y - z)$$

$$y = \frac{1}{8}(24 - 2x + z)$$

$$z = \frac{1}{10}(58 - 3x - 4y)$$

1st iteration

$$x = \frac{1}{12}(31 - 0 - 0) = 2.5833$$

$$y = \frac{1}{8}(24 - 2(2.5833) + (0)) = 2.3542$$

$$z = \frac{1}{10}(58 - 3(2.5833) - 4(2.3542)) = 4.0833$$

2nd iteration

$$x = \frac{1}{12}(31 - 2.3542 - 4.0833) = 2.0469$$

$$y = \frac{1}{8}(24 - 2(2.0469) + 4.0833) = 2.9987$$

$$z = \frac{1}{10}(58 - 3(2.0469) - 4(2.9987)) = 3.9865$$

Proceeding in the same manner and writing the solutions in the tabular form we get

i	x	y	z
1	2.5833	2.3542	4.0833
2	2.0469	2.9987	3.9865
3	2.0012	2.998	4.0004
4	2.0001	3	4
5	2	3	4
6	2	3	4

Since the values in 5th and 6th iteration are same, the solution is $x = 2, y = 3, z = 4$ up to four decimal places.

Solve by Gauss Seidel method, the system

$$9x - y + 2z = 9, x + 10y - 2z = 15, 2x - 2y - 13z = -17$$

Solution:

$$|-1| + |2| = 3 < |9|, |1| + |-2| < |10|, |2| + |-2| < |-13|$$

Therefore the given equations are diagonally dominant, so we can use Gauss Seidel method.

$$x = \frac{1}{9}(9 + y - 2z)$$

$$y = \frac{1}{10}(15 - x + 2z)$$

$$z = \frac{1}{13}(17 + 2x - 2y)$$

1st iteration

$$x^{(1)} = \frac{1}{9}\big(9 + 0 - 2(0)\big) = 1$$

$$y^{(1)} = \frac{1}{10}\big(15 - 1 + 2(0)\big) = 1.4$$

$$z^{(1)} = \frac{1}{13}(17 + 2(1) - 2(1.4)) = 1.2462$$

2nd iteration

$$x^{(2)} = \frac{1}{9}\big(9 + 1.4 - 2(1.2462)\big) = 0.8786$$

$$y^{(2)} = \frac{1}{10}\big(15 - 0.8786 + 2(1.2462)\big) = 1.6614$$

$$z^{(2)} = \frac{1}{13}(17 + 2(0.8786) - 2(1.6614)) = 1.1873$$

Proceeding in the same manner and writing the solutions in the tabular form we get

i	x	y	z
1	1	1.4	1.2462
2	0.8786	1.6614	1.1873
3	0.9208	1.6454	1.1962
4	0.917	1.6475	1.1953
5	0.9174	1.6473	1.1954
6	0.9174	1.6473	1.1954

Since the values in 5th and 6th iteration are same, the solution is $x = 0.9174, y = 1.6473, z = 1.1954$ up to four decimal places.

Gauss Elimination method

Solve by Gauss Elimination method, the system

$$12x + y + z = 31, 2x + 8y - z = 24, 3x + 4y + 10z = 58$$

Solution:

$$[A:B] = \begin{bmatrix} 12 & 1 & 1 & | & 31 \\ 2 & 8 & -1 & | & 24 \\ 3 & 4 & 10 & | & 58 \end{bmatrix}$$

$$\sim \begin{bmatrix} 12 & 1 & 1 & | & 31 \\ 0 & 7.83 & -1.17 & | & 18.83 \\ 0 & 3.75 & 9.75 & | & 50.25 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - \frac{2}{12} R_1 \\ R_3 \to R_3 - \frac{3}{12} R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 12 & 1 & 1 & | & 31 \\ 0 & 7.83 & -1.17 & | & 18.83 \\ 0 & 0 & 10.31 & | & 41.24 \end{bmatrix} \quad R_3 \to R_3 - \frac{3.75}{7.83} R_2$$

$$12x + y + z = 31$$

$$7.83y - 1.17z = 18.83$$

$$10.31z = 41.24$$

$$z = \frac{41.24}{10.31} = 4$$

$$7.83y - 1.17(4) = 18.83 \Rightarrow y = \frac{18.83 + 1.17(4)}{7.83} = 3$$

$$12x + 3 + 4 = 31 \Rightarrow x = \frac{31 - 7}{12} = 2$$

Therefore the solution is $x = 2, y = 3, z = 4$

Solve by Gauss Elimination method, the system

$$9x - y + 2z = 9, x + 10y - 2z = 15, 2x - 2y - 13z = -17$$

Solution:

$$[A:B] = \begin{bmatrix} 9 & -1 & 2 & | & 9 \\ 1 & 10 & -2 & | & 15 \\ 2 & -2 & -13 & | & -17 \end{bmatrix}$$

$$\sim\left[\begin{array}{ccc|c} 9 & -1 & 2 & 9 \\ 0 & 10.1111 & -2.2222 & 14 \\ 0 & -1.7778 & -13.4444 & -19 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{1}{9}R_1 \\ R_3 \to R_3 - \frac{2}{9}R_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 9 & -1 & 2 & 9 \\ 0 & 10.1111 & -2.2222 & 14 \\ 0 & 0 & -13.8352 & -16.5385 \end{array}\right] \quad R_3 \to R_3 - \frac{-1.7778}{10.1111}R_2$$

$$9x - y + 2z = 9$$

$$10.1111y - 2.2222z = 14$$

$$-13.8352z = 41.24 \Rightarrow z = \frac{-16.5385}{-13.8352} = 1.1954$$

$$10.1111y - 2.2222(1.1954) = 14 \Rightarrow y = \frac{14 + 2.2222(1.1954)}{10.1111} = 1.6473$$

$$9x - 1.6473 + 2(1.1954) = 9 \Rightarrow x = \frac{9 + 1.6473 - 2(1.1954)}{9} = 0.9174$$

Therefore the solution is $x = 0.9174, y = 1.6473, z = 1.1954$

Gauss Jordon method

Solve by Gauss Jordon method, the system

$$12x + y + z = 31, 2x + 8y - z = 24, 3x + 4y + 10z = 58$$

Solution:

$$[A:B] = \left[\begin{array}{ccc|c} 12 & 1 & 1 & 31 \\ 2 & 8 & -1 & 24 \\ 3 & 4 & 10 & 58 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 12 & 1 & 1 & 31 \\ 0 & 7.83 & -1.17 & 18.83 \\ 0 & 3.75 & 9.75 & 50.25 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - \frac{2}{12}R_1 \\ R_3 \to R_3 - \frac{3}{12}R_1 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 12 & 0 & 1.15 & 28.6 \\ 0 & 7.83 & -1.17 & 18.83 \\ 0 & 0 & 10.31 & 41.24 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{1}{7.83}R_2 \\ R_3 \to R_3 - \frac{3.75}{7.83}R_2 \end{array}$$

$$\sim\left[\begin{array}{ccc|c} 12 & 0 & 0 & 24 \\ 0 & 7.83 & 0 & 23.51 \\ 0 & 0 & 10.31 & 41.24 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - \frac{1.15}{10.31}R_3 \\ R_2 \to R_2 - \frac{-1.17}{10.31}R_3 \end{array}$$

$$12x = 24 \Rightarrow x = \frac{24}{12} = 2$$

$$7.83y = 23.51 \Rightarrow y = \frac{23.51}{7.83} = 3$$

$$10.31z = 41.24 \Rightarrow z = \frac{41.24}{10.31} = 4$$

Therefore the solution is $x = 2, y = 3, z = 4$

Solve by Gauss Jordon method, the system

$$9x - y + 2z = 9, x + 10y - 2z = 15, 2x - 2y - 13z = -17$$

Solution:

$$[A:B] = \left[\begin{matrix} 9 & -1 & 2 \\ 1 & 10 & -2 \\ 2 & -2 & -13 \end{matrix} \left| \begin{matrix} 9 \\ 15 \\ -17 \end{matrix} \right.\right]$$

$$\sim \left[\begin{matrix} 9 & -1 & 2 \\ 0 & 10.1111 & -2.2222 \\ 0 & -1.7778 & -13.4444 \end{matrix} \left| \begin{matrix} 9 \\ 14 \\ -19 \end{matrix} \right.\right] \quad \begin{matrix} R_2 \to R_2 - \frac{1}{9} R_1 \\ R_3 \to R_3 - \frac{2}{9} R_1 \end{matrix}$$

$$\sim \left[\begin{matrix} 9 & 0 & 1.7802 \\ 0 & 10.1111 & -2.2222 \\ 0 & 0 & -13.8352 \end{matrix} \left| \begin{matrix} 10.3846 \\ 14 \\ -16.5385 \end{matrix} \right.\right] \quad \begin{matrix} R_1 \to R_1 - \frac{-1}{10.1111} R_2 \\ R_3 \to R_3 - \frac{-1.7778}{10.1111} R_2 \end{matrix}$$

$$\sim \left[\begin{matrix} 9 & 0 & 0 \\ 0 & 10.1111 & 0 \\ 0 & 0 & -13.8352 \end{matrix} \left| \begin{matrix} 8.2566 \\ 16.6564 \\ -16.5385 \end{matrix} \right.\right] \quad \begin{matrix} R_1 \to R_1 - \frac{1.7802}{-13.8352} R_3 \\ R_2 \to R_2 - \frac{-2.2222}{-13.8352} R_3 \end{matrix}$$

$$9x = 8.2566 \Rightarrow x = \frac{8.2566}{9} = 0.9174$$

$$10.1111y = 16.6564 \Rightarrow y = \frac{16.6564}{10.1111} = 1.6473$$

$$-13.8352z = -16.5385 \Rightarrow z = \frac{-16.5385}{-13.8352} = 1.1954$$

Therefore the solution is $x = 0.9174, y = 1.6473, z = 1.1954$

NEWTON'S METHOD OR NEWTON-RAPHSON METHOD

Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton- Raphson method.

Solution :

Let $f(x) = x^4 - x - 10 \ = 0.$

Now, $f(0) = (0)^4 - (0) - 10 = -10 \quad (-ve)$

$f(1) = (1)^4 - (1) - 10 = -10 \quad (-ve)$

$f(2) = (2)^4 - (2) - 10 = +4 \quad (+ve)$

Therefore the root lies between 1 & 2.

Let us take $x_0 = 2 \ \{Near\ to\ zero\}$.

The Newton- Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \dots (1) .$$

$$Let \qquad f(x) = x^4 - x - 10 \quad \text{and} \quad f^1(x) = 4\,x^3 - 1$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right].$$

$$x_1 = 2 - \left[\frac{f(2)}{f'(2)}\right].$$

$$x_1 = 2 - \left[\frac{(2)^4 - (2) - 10}{4\,(2)^3 - 1}\right].$$

$$x_1 = 2 - \left[\frac{4}{31}\right].$$

$$x_1 = 1.8709 .$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right].$$

$$x_2 = 2 - \left[\frac{f(1.8709\,)}{f'(1.8709\,)}\right].$$

$$x_2 = 1.8709 \ - \left[\frac{(1.8709\,)^4 - (1.8709\,) - 10}{4\,(1.8709\,)^3 - 1}\right].$$

$$x_2 = 1.8709 \ - \left[\frac{0.3835}{25.199}\right].$$

$$x_2 = 1.856 .$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right].$$

$$x_3 = 1.856 - \left[\frac{f(1.856\,)}{f'\,(1.856\,)}\right].$$

$$x_3 = 1.856 \; - \left[\frac{(1.856\,)^4 - (1.856\,) - 10}{4\,(1.856)^3 - 1}\right].$$

$$x_3 = 1.856 \; - \left[\frac{0.010}{24.574}\right].$$

$$x_3 = 1.856\,.$$

The root of the equation $x^4 - x = 10$ is 1.856.

Using Newton's iterative method to find the root between 0 and 1 of $x^3 = 6x - 4$ correct to three decimal places.

Solution :

Let $f(x) = x^3 - 6x + 4 \quad = 0.$

Now, $f(0) = (0)^3 - 6(0) + 4 = +4 \quad (+ve)$

$f(1) = (1)^3 - 6(1) + 4 = -1 \quad (-ve)$

Therefore the root lies between 0 & 1.

Let us take $x_0 = 1 \quad \{Near\ to\ zero\}$.

The Newton- Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\,(x_n)} \ldots (1)\,.$$

$$Let \qquad f(x) = x^3 - 6x + 4 \ \text{ and } \ f^1(x) = 3\,x^2 - 6$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'\,(x_0)}\right].$$

$$x_1 = 1 - \left[\frac{f(1)}{f'\,(1)}\right].$$

$$x_1 = 1 - \left[\frac{(1)^3 - 6(1) + 4}{3\,(1)^2 - 6}\right].$$

$$x_1 = 1 - \left[\frac{-1}{-3}\right].$$

$$x_1 = 0.666\,.$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'\,(x_1)}\right].$$

$$x_1 = 0.666 - \left[\frac{f(0.666)}{f'\,(0.666)}\right].$$

$$x_2 = 0.666 - \left[\frac{(0.666)^3 - 6(0.666) + 4}{3\,(0.666)^2 - 6}\right].$$

$$x_2 = 0.666 - \left[\frac{0.28}{-4.65}\right].$$

$$x_1 = 0.73\,.$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'\,(x_2)}\right].$$

$$x_3 = 0.73 - \left[\frac{f(0.73\,)}{f'\,(0.73\,)}\right].$$

$$x_3 = 0.73 - \left[\frac{(0.73)^3 - 6(0.73) + 4}{3\,(0.73)^2 - 6}\right].$$

$$x_3 = 0.73\ - \left[\frac{0.009}{-4.4013}\right].$$

$$x_3 = 0.7320\,.$$

The root of the equation $x^3 - 6x + 4 = 0$ is 0.732.

Find the real positive root of $3x - \cos x - 1 = 0$ correct to six decimal places using Newton method.

Solution :

Let $f(x) = 3x - \cos x - 1 = 0.$

Now, $f(0) = 3(0) - \cos(0) - 1 = -2$ $(-ve)$

$f(1) = 3(1) - \cos(1) - 1 = 1.459698$ $(-ve)$

Therefore the root lies between 0 & 1.

Let us take $x_0 = 1$ $\{Near\ to\ zero\}$.

The Newton- Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\,(x_n)} \ldots (1)\,.$$

$$Let \qquad f(x) = 3x - \cos x - 1 \quad \text{and} \quad f^1(x) = 3 + \sin x$$

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'\,(x_0)}\right].$$

$$x_1 = 1 - \left[\frac{f(1)}{f'\,(1)}\right].$$

$$x_1 = 1 - \left[\frac{3(1) - \cos(1) - 1}{3 + \sin(1)}\right].$$

$$x_1 = 0.62002.$$

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right].$$

$$x_2 = 0.62002 - \left[\frac{f(0.62002)}{f'(0.62002)}\right].$$

$$x_2 = 0.62002 - \left[\frac{3(0.62002) - \cos(0.62002) - 1}{3 + sin\,(0.62002)}\right].$$

$$x_2 = 0.60712\,.$$

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right].$$

$$x_3 = 0.60712 - \left[\frac{f(0.60712)}{f'(0.60712)}\right].$$

$$x_3 = 0.60712 - \left[\frac{3(0.60712) - \cos(0.60712) - 1}{3 + sin\,(0.60712)}\right].$$

$$x_3 = 0.6071\,.$$

The root of the equation $x^4 - x = 10$ is 0.60712.

FIXED POINT ITERATION OR ITERATION METHOD

The condition for convergence of a method

Let $f(x) = 0$ be the given equation whose actual root is r. The equation $f(x) = 0$ be written as $x = g(x)$. Let I be the interval containing the root $x = r$. If $|g'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, x_2, \dots\dots.\, x_n$ will converge to r, if the initial starting value x_0 is chosen in I.

Note 1. Since $|x_n - r| \le K|x_{n-1} - r|$ where K is a constant the convergence is linear and the convergence is of order 1.

Note 2. The sufficient condition for the convergence is $|g'(x)| < 1$ for all x in I

Find the positive root of $x^2 - 2x - 3 = 0$ by Iteration method.

Solution :

Let $f(x) = x^2 - 2x - 3 = 0$.

Now, $f(0) = (0)^2 - 2(0) - 3 = -10 \quad (-ve)$

$$f(1) = (0)^2 - 2(0) - 3 = -10 \quad (-ve)$$
$$f(2) = (0)^2 - 2(0) - 3 = +4 \quad (+ve)$$

Therefore the root lies between 1 & 2.

Let us take $x_0 = 2$ $\{Near\ to\ zero\}$.

$$x^2 - 2x - 3 = 0 \Rightarrow x^2 = 2x + 3$$
$$\Rightarrow x = \sqrt{2x + 3}$$
$$\Rightarrow x = g(x) = \sqrt{2x + 3}$$

Let $x_0 = 2$

$$x_1 = g(x_0) = \sqrt{2x_0 + 3} = \sqrt{2(2) + 3} = 2.6457$$
$$x_2 = g(x_1) = \sqrt{2x_1 + 3} = \sqrt{2(2.6457) + 3} = 2.8795$$

$$x_3 = g(x_2) = \sqrt{2x_2 + 3} = \sqrt{2(2.8795) + 3} = 2.9595$$
$$x_4 = g(x_3) = \sqrt{2x_3 + 3} = \sqrt{2(2.9595) + 3} = 2.9864$$
$$x_5 = g(x_4) = \sqrt{2x_4 + 3} = \sqrt{2(2.9864) + 3} = 2.99549$$
$$x_6 = g(x_5) = \sqrt{2x_5 + 3} = \sqrt{2(2.99549) + 3} = 2.9985$$
$$x_7 = g(x_6) = \sqrt{2x_6 + 3} = \sqrt{2(2.9985) + 3} = 2.9995$$
$$x_8 = g(x_7) = \sqrt{2x_7 + 3} = \sqrt{2(2.9995) + 3} = 2.9998$$
$$x_9 = g(x_8) = \sqrt{2x_8 + 3} = \sqrt{2(2.9998) + 3} = 2.9999$$
$$x_{10} = g(x_9) = \sqrt{2x_9 + 3} = \sqrt{2(2.9999) + 3} = 2.9999$$

Hence the root of the equation is $x^2 - 2x - 3 = 0$ is 2.9999.

Find the Real root of the equation $\boldsymbol{x^3 + x^2 - 100}$ by Fixed point iteration method.

Solution:

$$\text{Let} \quad f(x) = x^3 + x^2 - 100 = 0.$$
$$f(0) = (0)^3 + (0)^2 - 100 = -100 \quad (-ve).$$
$$f(1) = (1)^3 + (1)^2 - 100 = -98 \quad (-ve).$$
$$f(2) = (2)^3 + (2)^2 - 100 = -88 \quad (-ve).$$
$$f(3) = (3)^3 + (3)^2 - 100 = -64 \quad (-ve).$$
$$f(4) = (4)^3 + (4)^2 - 100 = -20 \quad (-ve).$$
$$f(5) = (5)^3 + (5)^2 - 100 = +50 \quad (+ve).$$

The root lies between 4 & 5.

$$Since \qquad x^3 + x^2 - 100 = 0$$

$$\Rightarrow \quad x^2(x+1) = 100$$

$$\Rightarrow \quad x^2 = \frac{100}{(x+1)}$$

$$\Rightarrow \quad x = g(x) = \frac{10}{\sqrt{x+1}} = 10\,[x+1]^{\frac{1}{2}}$$

Now, $g^{'}(x) = 10\left(\frac{1}{2}\right)\,[x+1]^{\left(-\frac{3}{2}\right)} = 5\,[x+1]^{\left(-\frac{3}{2}\right)}$

$$g^{'}(4) = 5\,[4+1]^{\left(-\frac{3}{2}\right)} = \; < 1$$

$$g^{'}(5) = 5\,[5+1]^{\left(-\frac{3}{2}\right)} = \; < 1$$

So that we can use the iteration method.

Let $x_0 = 4$

$$x_1 = g(x_0) = \frac{10}{\sqrt{x_0+1}} = \frac{10}{\sqrt{4\,+1}} = \frac{10}{2.236} = 4.4721$$

$$x_2 = g(x_1) = \frac{10}{\sqrt{x_1+1}} = \frac{10}{\sqrt{4.4721\,+1}} = \frac{10}{2.1147} = 4.2748$$

$$x_3 = g(x_2) = \frac{10}{\sqrt{x_2+1}} = \frac{10}{\sqrt{4.2748+1}} = 4.3541$$

$$x_4 = g(x_3) = \frac{10}{\sqrt{x_3+1}} = \frac{10}{\sqrt{4.3541+1}} = 4.3217$$

$$x_5 = g(x_4) = \frac{10}{\sqrt{x_4+1}} = \frac{10}{\sqrt{4.3217+1}} = 4.3348$$

$$x_6 = g(x_5) = \frac{10}{\sqrt{x_5+1}} = \frac{10}{\sqrt{4.3348+1}} = 4.3295$$

$$x_7 = g(x_6) = \frac{10}{\sqrt{x_6+1}} = \frac{10}{\sqrt{4.3295\,+1}} = 4.3316$$

$$x_8 = g(x_7) = \frac{10}{\sqrt{x_7+1}} = \frac{10}{\sqrt{4.3316\,+1}} = 4.3307$$

$$x_9 = g(x_8) = \frac{10}{\sqrt{x_8+1}} = \frac{10}{\sqrt{4.3307+1}} = 4.3311$$

$$x_{10} = g(x_9) = \frac{10}{\sqrt{x_9+1}} = \frac{10}{\sqrt{4.3311+1}} = 4.3310$$

$$x_{11} = g(x_{10}) = \frac{10}{\sqrt{x_{10}+1}} = \frac{10}{\sqrt{4.3310+1}} = 4.3310$$

Hence the root of the equation is $x^3 + x^2 - 100 = 0$ is 4.3310.

Find the real root of the equation $\cos x = 3x - 1$ correct to five decimal places using fixed point iteration method.

Solution:

$$\text{Let} \quad f(x) = \cos x - 3x + 1 = 0.$$

$$f(0) = \cos(0) - 3(0) + 1 = 2 \qquad (+ve).$$

$$f(1) = \cos(1) - 3(1) + 1 = -1.4597 \quad (+ve).$$

The root lies between 0 & 1.

$$Since \qquad \cos x - 3x + 1 = 0$$

$$\Rightarrow \; 3x = \cos x + 1$$

$$\Rightarrow \quad x = \frac{1}{3}(1 + \cos x)$$

$$\Rightarrow \quad x = g(x) = \frac{1}{3}(1 + \cos x)$$

Now, $g'(x) = \frac{1}{3}(-\sin x) = -\frac{1}{3}\sin x$

$$g'(0) = -\frac{1}{3}\sin(0) = \; 0 \quad < 1$$

$$g'(1) = -\frac{1}{3}\sin(1) = \; 0.284 < 1$$

So that we can use the iteration method.

Let $x_0 = 4$

$$x_1 = g(x_0) = \frac{1}{3}(1 + \cos x_0) = \frac{1}{3}(1 + (-0.6536)\;) = 0.11545$$

$$x_2 = g(x_1) = \frac{1}{3}(1 + \cos x_1) = \frac{1}{3}(1 + \cos(0.11545)\;) = 0.6644$$

$$x_3 = g(x_2) = \frac{1}{3}(1 + \cos x_2) = \frac{1}{3}(1 + \cos[0.6644]\;) = 0.5957$$

$$x_4 = g(x_3) = \frac{1}{3}(1 + \cos x_3) = 0.6092$$

$$x_5 = g(x_4) = \frac{1}{3}(1 + \cos x_4) = 0.60669$$

$$x_6 = g(x_5) = \frac{1}{3}(1 + \cos x_5) = 0.60717$$

$$x_7 = g(x_6) = \frac{1}{3}(1 + \cos x_6) = 0.60708$$

$$x_8 = g(x_7) = \frac{1}{3}(1 + \cos x_7) = 0.60710$$

$$x_9 = g(x_8) = \frac{1}{3}(1 + \cos x_8) = 0.60710$$

Hence the root of the equation is $\cos x = 3x - 1$ is 0.60710.

Triangularisation method or LU decomposition method 0r LU factorization method

Solve the following system by triangularisation method

$$\boldsymbol{x + y + z = 1, 4x + 3y - z = 6, 3x + 5y + 3z = 4}$$

Solution:

The given system is equivalent to

$$AX = B \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

$$Let\ L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Let $LU = A$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

Equating the coefficients on both sides we get,

$$u_{11} = 1, u_{12} = 1, u_{13} = 1$$

$$l_{21}u_{11} = 4 \Rightarrow l_{21}(1) = 4 \Rightarrow l_{21} = 4$$

$$l_{21}u_{12} + u_{22} = 3 \Rightarrow (4)(1) + u_{22} = 3 \Rightarrow u_{22} = -1$$

$$l_{21}u_{13} + u_{23} = (4)(1) + u_{23} = -1 \Rightarrow u_{23} = -1 - 4 = -5 \Rightarrow u_{23} = -5$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31}(1) = 3 \Rightarrow l_{31} = 3$$

$$l_{31}u_{12} + l_{32}u_{22} = 5 \Rightarrow (3)(1) + l_{32}(-1) = 5 \Rightarrow l_{32} = 3 - 5 \Rightarrow l_{32} = -2$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 3 \Rightarrow (3)(1) + (-2)(-5) + u_{33} = 3 \Rightarrow u_{33} = 3 - 3 - 10 \Rightarrow u_{33} = -10$$

$$AX = B \Rightarrow \ LUX = B \Rightarrow \ LY = B \ \ where \ UX = Y$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

By forward substitution method we get

$$y_1 = 1$$

$$4y_1 + y_2 = 6 \Rightarrow 4(1) + y_2 = 6 \Rightarrow y_2 = 2$$

$$3y_1 - 2y_2 + y_3 = 4 \Rightarrow 3(1) - 2(2) + y_3 = 4 \Rightarrow y_3 = 5$$

$$UX = Y$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

By backward substitution method we get

$$-10z = 5 \Rightarrow \ z = -\frac{1}{2}$$

$$-y - 5z = 2 \Rightarrow -y - 5\left(-\frac{1}{2}\right) = 2 \Rightarrow y = \frac{5}{2} - 2 \Rightarrow y = \frac{1}{2}$$

$$x + y + z = 1 \Rightarrow x + \frac{1}{2} - \frac{1}{2} = 1 \Rightarrow x = 1$$

$$\text{Hence } x = 1, y = \frac{1}{2}, z = -\frac{1}{2}.$$

Solve the following system by triangularisation method

$$\boldsymbol{x + 5y + z = 14, 2x + y + 3z = 13, 3x + y + 4z = 17}$$

Solution:

The given system is equivalent to

$$AX = B \Rightarrow \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

$$Let \ L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

Let $LU = A$

$$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Equating the coefficients on both sides we get,

$$u_{11} = 1, u_{12} = 5, u_{13} = 1$$

$$l_{21}u_{11} = 2 \Rightarrow l_{21}(1) = 2 \Rightarrow l_{21} = 2$$

$$l_{21}u_{12} + u_{22} = 1 \Rightarrow (2)(5) + u_{22} = 1 \Rightarrow u_{22} = -9$$

$$l_{21}u_{13} + u_{23} = 3 \Rightarrow (2)(1) + u_{23} = 3 \Rightarrow u_{23} = 1$$

$$l_{31}u_{11} = 3 \Rightarrow l_{31}(1) = 3 \Rightarrow l_{31} = 3$$

$$l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow (3)(5) + l_{32}(-9) = 1 \Rightarrow l_{32} = \frac{14}{9}$$

$$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 4 \Rightarrow (3)(1) + \left(\frac{14}{9}\right)(1) + u_{33} = 4 \Rightarrow u_{33} = 1 - \frac{14}{9} \Rightarrow u_{33} = -\frac{5}{9}$$

$$AX = B \Rightarrow \ LUX = B \Rightarrow LY = B \ \ where\ UX = Y$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

By forward substitution method we get

$$y_1 = 14$$

$$2y_1 + y_2 = 13 \Rightarrow 2(14) + y_2 = 13 \Rightarrow y_2 = -15$$

$$3y_1 + \frac{14}{9}y_2 + y_3 = 17 \Rightarrow 3(14) + \frac{14}{9}(-15) + y_3 = 17 \Rightarrow y_3 = -\frac{5}{3}$$

$$UX = Y$$

$$\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{pmatrix}$$

By backward substitution method we get

$$-\frac{5}{9}z = -\frac{5}{3} \Rightarrow z = 3$$

$$-9y + z = -15 \Rightarrow -9y + 3 = -15 \Rightarrow -9y = -18 \Rightarrow y = 2$$

$$x + 5y + z = 14 \Rightarrow x + 5(2) + 3 = 14 \Rightarrow x = 1$$

Hence $x = 1, y = 2, z = 3$.